Theorem 1.1. The only solution to the differential equation \( \frac{dy}{dx} = ky \) is the function
\[
y = Ce^{kx}
\]

Theorem 1.2. Newton's Law of Cooling states (using \( T(0) \), \( T_s \))
\[
T = T_s + (T_0 - T_s)e^{-kt}
\]

Definition 1.3. The linearization \( L(x) \) of \( f(x) \) at \( a \) is defined by
\[
L(x) = f(a) + f'(a)(x-a)
\]

Example 1.4. A scientist is researching a newly-discovered species of bacteria that grows at a rate proportional to its size. Two hours after putting a culture of 100 cells into a favourable growth medium, she measures 450 cells. Find the function describing the number of bacteria at time \( t \).

\[
\begin{align*}
(2, 450) &\quad C = 100 \\
450 &= 100e^{2k} \\
\frac{\ln(4.5)}{2} &= k \\
4.5 &= e^{2k} \\
ln 4.5 &= 2k \\
\end{align*}
\]

Example 1.5. Use a linear approximation to estimate \((1.999)^4\).

\[
\begin{align*}
f(0) &= x^4 & y &= 16 + 32(x - 2) \\
f(2) &= 16 \\
f'(x) &= 4x^3 & y &= 16 + 32(1.999 - 2) \\
f'(2) &= 32 & y &= 16 + 32(-0.001) \\
\end{align*}
\]

\[
y \approx 15.068
\]
\[
f(1.999) \approx 15.068
\]
Example 2.1. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

\[
\text{Find } \frac{ds}{dt} \text{ when } y = 4 \qquad \text{Know } \frac{dx}{dt} = 1.6 \quad \frac{dy}{dt} = -1.6
\]

\[
\begin{align*}
S &= 2y \\
X \frac{ds}{dt} + S \frac{dx}{dt} &= 0 \\
\frac{ds}{dt} &= -\frac{S}{X} \frac{dx}{dt} \\
\frac{ds}{dt} &= -\frac{3}{8} \cdot 1.6 \implies \left(\frac{8}{3}\right) \\
\frac{ds}{dt} &= -\frac{3}{8} \cdot \frac{8}{5} \implies -\frac{24}{40} \implies \boxed{\frac{ds}{dt} = -\frac{3}{5}}
\end{align*}
\]

Example 2.2. Find the dimensions of a rectangle with area 1000 square metres whose perimeter is as small as possible.

\[
\begin{align*}
xy &= 1000 \\
y &= \frac{1000}{x} \\
x &= \sqrt{1000} \\
y &= \sqrt{\frac{1000}{x}} \\
\end{align*}
\]

\[
\begin{align*}
2x + 2y &= P \\
2x + \frac{2000}{x} &= P \\
\frac{2}{x^2} &= P \\
\frac{2}{x^2} &= 0 \\
2x^2 &= 2000 \\
x &= \sqrt{1000} \\
x &= \sqrt{1000} \\
x &= +\sqrt{1000}
\end{align*}
\]

When \( x = \sqrt{1000} \), there is a local minimum, but since it is the only critical point, it is also an absolute minimum.

The derivative is also undefined at \( x = 0 \); however, this critical point is irrelevant since when \( x = 0 \) there is no rectangle.
Math 221 AD2: Test 1 Review—Extreme Values and Concavity
October 18, 2018

Example 3.1. Let \( f(x) = x^3 - 6x^2 + 5 \). Find any absolute maximum and absolute minimum of \( f \) on \([-3, 5]\). Find any local maximums and local minimums of \( f \) on \([-3, 5]\). State the subintervals of \([-3, 5]\) on which \( f \) is increasing and those on which \( f \) is decreasing.

\[
\begin{align*}
\mathcal{f}(x) &= x^3 - 6x^2 + 5 \quad \text{[-3, 5]} \\
\mathcal{f}'(x) &= 3x^2 - 12x \\
&= 3x(x-4) \\
x &= 4, 0 \\
\end{align*}
\]

\( f(-3) = -76 \) abs min
\( f(0) = 5 \) abs max
\( f(4) = -27 \) local min
\( f(5) = -20 \)

\[
\begin{align*}
\mathcal{f}''(x) &= -12x + 12 \\
&= -12(x-1) \\
- \text{increasing} \ (-3, 0) \cup (4, 5) \\
- \text{decreasing} \ (0, 4)
\end{align*}
\]

Example 3.2. Let \( f(x) = 2x^3 - 9x^2 + 12x - 3 \). Find the local maximum and minimum values of \( f \) using the Second Derivative Test. Find the intervals of concavity and the inflection points.

\[
\begin{align*}
\mathcal{f}(x) &= 2x^3 - 9x^2 + 12x - 3 \\
\mathcal{f}'(x) &= 6x^2 - 18x + 12 \\
\mathcal{f}''(x) &= 12x - 18 \\
x &= \frac{3}{2} \rightarrow \text{possible inf. point} \\
\mathcal{f}'''(x) &= 12 \\
&= 12 \\
@ x = 1 \text{ local max} \\
@ x = 2 \text{ local min}
\end{align*}
\]
Theorem 4.1 (Mean Value Theorem).

If \( f(x) \) is continuous on \( [a, b] \) and if \( f(x) \) is differentiable on \( (a, b) \) then there is a point \( C \) in \( (a, b) \) where \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

Theorem 4.2 (Rolle’s Theorem).

If \( f(x) \) is continuous on a closed interval \( [a, b] \), and differentiable on an open interval \( (a, b) \), and \( f(a) = f(b) \), then there is a point \( C \) at which \( f'(c) = 0 \).

Example 4.3. Let \( f(x) = x^3 - 3x + 2 \). Verify that \( f \) satisfies the hypotheses of the Mean Value Theorem on \([-2, 2]\) and then find all numbers \( c \) that satisfy the conclusion of the theorem.

\(- f'(x) \) is continuous on the interval because it’s a polynomial.

\(- f'(x) \) is differentiable on the interval because it’s a polynomial.

\( f(x) = x^3 - 3x + 2 \) \( f(-2) = 0 \) \( f'(c) = \frac{4 - 6}{4} = -1 \)

\( f''(x) = 3x^2 - 3 \) \( f'(2) = 4 \)

\( f'(c) = 1 = 3c^2 - 3 \)

\( c = \pm \sqrt{\frac{4}{3}} \)

Example 4.4. Let \( f(x) = x^3 + 2x \). Show that \( f \) has at most one root. What is it?

Suppose \( f(x) \) has 2 roots \( a \& b \) with \( a < b \).

\( f(a) = f(b) = 0 \)

\( f \) is differentiable everywhere b/c it’s a polynomial

So apply Rolle’s Thm: get \( c \) in \((a, b)\)

\( w/ f'(c) = 0 \)

\( 3c^2 + 2 = 0 \)

There can NOT be such a real number \( c \), so \( f \) has at most one root.
Math 221 AD2: Test 1 Review—Indeterminate Forms and L'Hôpital's Rule
October 18, 2018

**Definition 5.1.** The indeterminate forms are:

\[ 0^0, 1^0, \infty \cdot 0, 0 \cdot \infty, \frac{0}{0}, \frac{\infty}{\infty} \]

**Theorem 5.2** (L'Hôpital's Rule).

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ when } \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \]

where \( f(x) \) and \( g(x) \) are differentiable functions and \( g(x) \neq 0 \).

**Example 5.3.** Find the limit.

\[ \lim_{x \to 0} \frac{x^2}{1 - \cos x} = \frac{0}{0} = \frac{2}{1} \]

\[ = 2 \]

---

**Example 5.4.** Find the limit.

\[ \lim_{x \to \infty} x e^{-x} = \lim_{x \to \infty} \frac{\ln x}{e^x} = \lim_{x \to \infty} e^{-x} \ln x \]

\[ = \lim_{x \to \infty} \frac{\ln x}{e^x} = \frac{\infty}{\infty} \]

\[ \frac{1}{e^x} = \frac{0}{\infty} = 0 \]

\[ \Rightarrow e^0 = 1 \]

*don't forget \( e^0 \!*.}