Math 221 CD1: Test 2 Review—Related Rates, Linear Approximations, and Newton’s Method
November 1, 2017

Definition 1.1. The linearization \( L(x) \) of \( f(x) \) at \( a \) is defined by

\[
L(x) = f(a) + f'(a)(x - a)
\]

Definition 1.2. Newton’s method is a technique to approximate a root (or zero) of \( f(x) \). A first guess \( x_1 \) is made, and then it proceeds recursively according to

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

Example 1.1. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

Given: \( \frac{dx}{dt} = -1.6 \text{ m/s} \)

Want: \( \frac{dy}{dt} \bigg|_{x=4} \)

\[
12y - xy = 24
\]

\[
y = \frac{24}{12-x}
\]

\[
12 \frac{dy}{dt} - (\frac{dx}{dt})y + \frac{dy}{dt}x = 0
\]

\[
12 \frac{dy}{dt} - (-1.6) \left( \frac{24}{12-x} \right) + \frac{dy}{dt} \left( 4 \right) = 0
\]

\[
12 \frac{dy}{dt} - (-1.6) (4) + \frac{dy}{dt} (4) = 0
\]

\[
12 \frac{dy}{dt} + 4.8 - 4 \frac{dy}{dt} = 0
\]

\[
8 \frac{dy}{dt} = -4.8
\]

\[
\frac{dy}{dt} = -0.6 \text{ m/s}
\]
Example 1.2. Use a linear approximation to estimate \((1.999)^4\).

\[ f(x) = x^4 \quad a = 2 \]

\[ L(x) = f(a) + f'(a)(x - a) \]

\[ L(x) = 16 + 32(1.999 - 2) \]

\[ (1.999)^4 \approx 16 + 32(-0.001) \]

\[ 16 - 0.032 \]

\[ 15.968 \]

Example 1.3. Estimate the \(x\)-value for the point of intersection on the graphs of \(y = x^3 + 2x\) and \(y = 2x + 4\) using Newton’s Method with an initial estimate of \(x_1 = 1\). You should use this method 2 times in order to obtain estimates \(x_2\) and \(x_3\).

\[ y = x^3 + 2x \quad y = 2x + 4 \quad x_1 = 1 \]

\[ x^3 + 2x = 2x + 4 \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

\[ f(x_n) = x^3 - 4 = 0 \]

\[ f'(x_n) = 3x^2 \]

\[ 1 - \frac{-3}{3} = 2 = x_2 \]

\[ 2 - \frac{4}{12} = \frac{5}{3} = x_3 \]
Math 221 CD1: Test 2 Review—Extreme Values, Concavity, and the Mean Value Theorem
November 1, 2017

**Theorem 3.1 (Mean Value Theorem).**
If a function is continuous on \([a, b]\) and differentiable on \((a, b)\) then there exists a point \(a < c < b\) that
\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

**Theorem 3.2 (Rolle’s Theorem).**
If a function is continuous on \([a, b]\) and is differentiable on \((a, b)\) and \(f(a) = f(b)\) then there exists a point \(c\) that \(a < c < b\) that \(f'(c) = 0\).

**Example 3.4.** Let \(f(x) = x^3 - 6x^2 + 5\). Does \(f\) have an absolute maximum and absolute minimum on \([-3, 5]\)? Why or why not? Find any absolute maximum and absolute minimum of \(f\) on \([-3, 5]\). Find any local maximums and local minimums of \(f\) on \([-3, 5]\). State the subintervals of \([-3, 5]\) on which \(f\) is increasing and those on which \(f\) is decreasing.

\[
f'(x) = 3x^2 - 12x
\]
\[
f'(x) = 3x^2 - 12x = 0
\]
\[
3x(x - 4) = 0
\]
\[
x = 0, 4
\]

\[
f'(-10) = 3(-10)^2 - 12(-10)
\]
\[
f'(2) = 3(2)^2 - 12(2)
\]
\[
f'(10) = 3(10)^2 - 12(10)
\]

On \([-3, 5]\) there exists an absolute maximum @ \((5, 100)\) and an absolute min. @ \((-3, -74)\) and a local minimum of \((4, -23)\) and a local maximum @ \((0, 5)\).

\[
f(0) = (0)^3 - 6(0)^2 + 5 = 5
\]
\[
f(4) = (4)^3 - 6(4)^2 + 5 = 64 - 92 + 5 = -23
\]
\[
f(-3) = (-3)^3 - 6(-3)^2 + 5 = -27 - 54 + 5 = -76
\]
\[
f(5) = (5)^3 - 6(5)^2 + 5 = 125 - 150 + 5 = 100
\]

\(f\) is increasing from \((-3, 0)\) \& \((4, 5)\) and is decreasing from \((0, 4)\).
Example 3.5. Let \( f(x) = 2x^3 - 9x^2 + 12x - 3 \). Find the local maximum and minimum values of \( f \) using the Second Derivative Test. Find the intervals of concavity and the inflection points.

\[
\frac{f(x)}{= 6x^2 - 18x + 12} \quad \frac{f'(x)}{= 12x^2 - 36x + 12} \quad \frac{f''(x)}{= 24x - 36}
\]

Max @ \( x = 1 \) \( 12x = 18 \) \( x = 1.5 \)
Min @ \( x = 2 \) \( x = 3/2 \)
Inflection @ \( x = 3/2 \)
Concave down : \((-\infty, 3/2)\)
Concave up : \((3/2, \infty)\)

Example 3.6. Let \( f(x) = x^3 - 3x + 2 \). Verify that \( f \) satisfies the hypotheses of the Mean Value Theorem on \([-2, 2]\) and then find all numbers \( c \) that satisfy the conclusion of the theorem.

- \( f \) is cont. on \([A, B]\) because it's a Polynomial
- \( f \) is diff. on \((A, B)\)

So there must be a value \( c \) between \((A, B)\) where

\[
f(c) = \frac{f(b) - f(a)}{b - a}
\]

\[
\frac{4 - 0}{2 - (-2)} = \frac{4}{4} = 1 \quad f'(c) = 1
\]

\[
f'(x) = 3x^2 - 3 = 1
\]

\[
\frac{\partial}{\partial x} x^2 = 4
\]

\[
x^2 = \frac{4}{3}
\]

\[
x = ±\sqrt{\frac{4}{3}}
\]
Math 221 CD1: Test 2 Review—Indeterminate Forms, L'Hôpital's Rule, and Optimization
November 1, 2017

**Definition 5.3.** The indeterminate forms are:

\[
\begin{array}{cccccc}
\frac{0}{0} & \pm \infty & 0 \cdot (\pm \infty) & 0^0 & 1^\infty & \infty^0 & \infty - \infty \\
\end{array}
\]

**Theorem 5.3 (L'Hôpital's Rule).**

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

provided 2nd limit exists or is \(\pm \infty\)

**Example 5.7.** Find the limit.

\[
\lim_{x \to 0} \frac{x^2}{1 - \cos x}
\]

**Indeterminate**

\[
L'H = \lim_{x \to 0} \frac{2x}{\sin x} \to 0 \text{ Indeterminate}
\]

\[
L'H = \lim_{x \to 0} \frac{2}{\cos x} = \frac{2}{1} = 2
\]

**Example 5.8.** Find the limit.

\[
\lim_{x \to \infty} x^{e^{-x}} = \lim_{x \to \infty} x = \lim_{x \to \infty} \frac{\ln(x)}{e^x}
\]

\[
= \lim_{x \to \infty} \frac{\ln(x)}{e^x} \to \infty \text{ Indeterminate}
\]

\[
L'H = \lim_{x \to \infty} \frac{1}{e^x} \to 0 = e^0 = 1
\]
Example 5.9. Find the dimensions of a rectangle with area 1000 square metres whose perimeter is as small as possible.

\[ A = xy = 1000 \]
\[ P = 2x + 2y \]
\[ x = \frac{1000}{y} \]
\[ P = 2 \left( \frac{1000}{y} \right) + 2y = 2000 \frac{1}{y} + 2y \]

\[ P' = -2000 \frac{1}{y^2} + 2 = 0 \]
\[ \frac{1}{y^2} = \frac{-2}{-2000} \]
\[ y^2 = 1000 \]
\[ y = \sqrt{1000} \]

\[ y = \sqrt{1000} \]
Math 221 CD1: Test 2 Review—Riemann Sums and Definite Integrals
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Theorem 7.4. If $f$ is continuous on $[a, b]$ or if $f$ has only a finite number of jump discontinuities on $[a, b]$, then $f$ is integrable on $[a, b]$.

Theorem 7.5. If $f$ is integrable on $[a, b]$, then the integral of $f$ from $a$ to $b$ can be computed using a limit of right Riemann sums. That is,

$$
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \Delta x
$$

where $\Delta x = \frac{b-a}{n}$ and $x_k = a + k\Delta x$

Example 7.10. If $\int_0^5 f(x) \, dx = 9$, $\int_0^2 g(x) \, dx = 6$, and $\int_2^5 g(x) \, dx = 2$, find

$$
\int_0^5 [2f(x) + 3g(x)] \, dx.
$$

$$
\int_0^5 2f(x) \, dx + \int_0^5 3g(x) \, dx
$$

$$
= 2 \int_0^5 f(x) \, dx + 3 \int_0^5 g(x) \, dx
$$

$$
= 2 \int_0^5 f(x) \, dx + 3 \left( \int_0^2 g(x) \, dx + \int_2^5 g(x) \, dx \right)
$$

$$
= 2 \cdot 9 + 3 \cdot (6 + 2)
$$

$$
= 18 + 24
$$

$$
= 42
$$
Example 7.11. Estimate the area under the graph of \( f(x) = 1 + x^2 \) from \( x = -1 \) to \( x = 2 \) using three rectangles and right endpoints. Sketch a picture illustrating this approximation. Then compute the correct area using a right Riemann sum.

![Diagram of rectangles and function graph](image)

\[
\lim_{n \to \infty} \frac{b - a}{n} \sum_{k=1}^{n} f\left( a + \frac{k(b-a)}{n} \right) = \frac{b-a}{n} \sum_{k=1}^{n} f\left( a + \frac{k(b-a)}{n} \right)
\]

\[
\frac{b-a}{n} = \frac{2 - (-1)}{3} = 1
\]

Rectangles will have width of 1

\[
1(f(0)) + 1(f(1)) + 1(f(2))
\]

\[
1 + 2 + 3 = \boxed{6}
\]

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \left( 1 + (-1 + \frac{k}{n})^2 \right) \frac{3}{n}
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( 1 + \frac{1}{n} \cdot \frac{k}{n} + \frac{9}{n^2} \right) \frac{3}{n}
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{6}{n} \sum_{k=1}^{n} \left( \frac{k}{n} - \frac{9}{n^2} k + \frac{27}{n^3} k^2 \right) \right)
\]

\[
= \lim_{n \to \infty} \left( \frac{6}{n} \cdot \frac{n(n+1)}{2} - \frac{9}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} \right)
\]

\[
= 6 + 9 + 9
\]

\[
= \boxed{24}
\]
Math 221 CD1: Test 2 Review—Antiderivatives and the Fundamental Theorem of Calculus  
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Definition 9.4. $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$

Theorem 9.6 (Fundamental Theorem of Calculus). Suppose:

1. $f$ is continuous on $[a, b]$ and let $g(x) = \int_{a}^{x} f(t) \, dt$
   Then
   1. $g$ is continuous on $[a, b]$
   2. $g$ is differentiable on $(a, b)$
   3. $g'(x) = f(x)$

2. $f$ is continuous on $[a, b]$

   \[
   \int_{a}^{b} f(x) \, dx = F(b) - F(a)
   \]

   where $F$ is any antiderivative of $f$ on $[a, b]$

Example 9.12. Find the derivative of the following function.

\[
g(x) = \int_{x}^{3} e^{t^2} + 3t^8 \, dt = \left. \int e^{t^2} + 3t^8 \, dt \right|_{x}^{3} = -e^{x^2} + 3x^8\]

\[
g'(x) = -e^{x^2} - 3x^8
\]
Example 9.13. Find the derivative of the following function.

\[
\int_0^{x^2 + \cosh(x)} \frac{1}{1 + u^2} + 2u - 5\sqrt{u} + 1 \, du
\]

\[
= \left[ \frac{1}{1 + (x^2 + \cosh(x))^2} \right] + 2 \left( \frac{x^2 + \cosh(x)}{x^2 + \cosh(x)} \right) - 5 \int \frac{\sqrt{x^2 + \cosh(x)}}{x^2 + \cosh(x) + 1} \, dx
\]

\[
= (2x + \sinh(x))
\]


\[
g(x) = \int_0^1 \frac{1}{x^2 + 1} + 2x^3 - \sqrt{x + \sec^2(x)} + e^2 \, dx
\]

antiderivative:

\[
\arctan(x) + \frac{1}{2}x^4 - \frac{2}{3}x^3 + \tan(x) + c \cdot e^x
\]

\[
= \arctan(1) + \frac{1}{2}(1)^4 - \frac{2}{3}(1)^3 + \tan(1) + e - \left( \arctan(0) + \frac{1}{2}(0)^4 - \frac{2}{3}(0)^3 + \tan(0) + e^0 \right)
\]

\[
= \frac{\pi}{4} + \frac{1}{2} - \frac{2}{3} + \tan(1) + c - \left( 0 - 0 - 0 - 1 \right)
\]

\[
= - \frac{\pi}{4} - \frac{2}{3} + \tan(1) + c
\]