Name: ____________________________________________________________

- Show **ALL** your work.
- There are **NO** calculators allowed on this exam.
- This exam has 12 questions.
1. Let \( f(x) = \frac{x - 2}{x^2 - 8x + 12} \). Circle the correct statement from the following.
   A. \( f(x) \) is differentiable at \( x = 2 \) because it is continuous at \( x = 2 \).
   B. \( f(x) \) is differentiable at \( x = 2 \) because \( f'(x) = \frac{-x^2 + 4x - 4}{(x^2 - 8x + 12)^2} \) for all \( x \).
   C. \( f(x) \) is not differentiable at \( x = 2 \) because \( \lim_{x \to 2^-} \frac{-x^2 + 4x - 4}{(x^2 - 8x + 12)^2} \neq \lim_{x \to 2^+} \frac{-x^2 + 4x - 4}{(x^2 - 8x + 12)^2} \).
   D. \( f(x) \) is not differentiable at \( x = 2 \) because it is not continuous at \( x = 2 \).

   **Solution:** D. \( f(x) \) is not differentiable at \( x = 2 \) because it is not continuous at \( x = 2 \).

2. Let \( f(x) = \frac{x^3 + 2x^2 - 7x + 3}{x^2 - 9} \). Circle the correct statement from the following.
   A. \( f(x) \) has a zero in \([0, 2]\) because \( f(0) < 0 \) and \( f(2) > 0 \).
   B. \( f(x) \) has no zeroes in \([0, 2]\) because \( f(0) < 0 \) and \( f(2) < 0 \).
   C. \( f(x) \) has two zeroes in \([0, 2]\).
   D. The Intermediate Value Theorem is inconclusive because \( f(x) \) is not continuous.

   **Solution:** C. \( f(x) \) has two zeroes in \([0, 2]\). (Consider \( x = 1 \).)
3. The function $f(x) = \sqrt[3]{\arctan\left(\frac{2x^7-3}{5y^7+2}\right)}$ is invertible. Find a formula for the inverse.

**Solution:** Exchange the role of $x$ and $y$ and solve for $y$ in terms of $x$:

\[
x = \sqrt[3]{\arctan\left(\frac{2y^7 - 3}{5y^7 + 2}\right)}
\]

\[
x^3 = \arctan\left(\frac{2y^7 - 3}{5y^7 + 2}\right)
\]

\[
\tan(x^3) = \frac{2y^7 - 3}{5y^7 + 2}
\]

\[
(5y^7 + 2)\tan(x^3) = 2y^7 - 3
\]

\[
5y^7\tan(x^3) - 2y^7 = -3 - 2\tan(x^3)
\]

\[
y^7(5\tan(x^3) - 2) = -3 - 2\tan(x^3)
\]

\[
y^7 = \frac{-3 - 2\tan(x^3)}{5\tan(x^3) - 2}
\]

\[
y = \left(\frac{-3 - 2\tan(x^3)}{5\tan(x^3) - 2}\right)^{1/7}
\]

which means that a formula for the inverse function is:

\[
f^{-1}(x) = \left(\frac{-3 - 2\tan(x^3)}{5\tan(x^3) - 2}\right)^{1/7}
\]
4. Evaluate the following limits without the use of derivatives. Show sufficient justification for each answer. An answer of ‘does not exist’ is not sufficient. For infinite limits you must state if it is $\infty$ or $-\infty$.

(a) \[ \lim_{x \to \infty} \frac{42 - 15e^{2x}}{6e^x + 100} \]

**Solution:** As $x \to \infty$ the numerator goes to $-\infty$ and the denominator to $+\infty$. Dividing by the dominant term $e^x$, the numerator still goes to $-\infty$ while the denominator goes to 6. Hence the limit is $-\infty$.

(b) \[ \lim_{x \to 5} \frac{x - 5}{\sqrt{42x + 46} - 16} \]

**Solution:**

\[
\lim_{x \to 5} \frac{x - 5}{\sqrt{42x + 46} - 16} \cdot \frac{\sqrt{42x + 46} + 16}{\sqrt{42x + 46} + 16} = \lim_{x \to 5} \frac{(x - 5)(\sqrt{42x + 46} + 16)}{42x + 46 - 256} = \lim_{x \to 5} \frac{(x - 5)(\sqrt{42x + 46} + 16)}{42(x - 5)} = \lim_{x \to 5} \frac{\sqrt{42x + 46} + 16}{42} = \frac{32}{42}
\]

(c) \[ \lim_{x \to \infty} \frac{\sin (x^4e^x - 3 \arctan (x))}{x^4} \]

**Solution:** $-1 \leq \sin (x^4e^x - 3 \arctan (x)) \leq 1$, so $-\frac{1}{x^4} \leq \frac{\sin (x^4e^x - 3 \arctan (x))}{x^4} \leq \frac{1}{x^4}$. Note that $\lim_{x \to \infty} -\frac{1}{x^4} = 0$ and $\lim_{x \to \infty} \frac{1}{x^4} = 0$, so by the squeeze theorem, we conclude that $\lim_{x \to \infty} \frac{\sin (x^4e^x - 3 \arctan (x))}{x^4} = 0$.

(d) \[ \lim_{x \to \infty} \frac{\ln (e^{100x} + e^{-100x})}{\ln (1 - e^{-x})} \]

**Solution:** As $x \to \infty$ the numerator goes to $+\infty$ and the denominator goes to 0. Note that $e^{-x} > 0$, so $1 - e^{-x} < 1$, so $\ln(1 - e^{-x}) < 0$. Hence the denominator approaches 0 from the left, so the limit is $-\infty$. 

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(e) \[ \lim_{x \to \ln 7} \frac{e^x - 7}{e^{2x} - 49} \]

**Solution:**

\[
\lim_{x \to \ln 7} \frac{e^x - 7}{e^{2x} - 49} = \lim_{x \to \ln 7} \frac{e^x - 7}{(e^x - 7)(e^x + 7)} = \lim_{x \to \ln 7} \frac{1}{e^x + 7} = \frac{1}{14}
\]

5. Consider the function \( f(x) = -90 + 30e^{\sqrt[3]{3}} \). Find the equation of the tangent line at the \( x \)-intercept of \( f \).

**Solution:** First, find the \( x \)-intercept by solving the following:

\[
0 = -90 + 30e^{\sqrt[3]{3}}
\]

\[
90 = 30e^{\sqrt[3]{3}}
\]

\[
3 = e^{\sqrt[3]{3}}
\]

\[
\ln(3) = \sqrt[3]{3}
\]

\[
\left( \ln(3) \right)^3 = x
\]

Thus, the \( x \)-intercept is the point \( (\ln(3))^3, 0 \). Now, take the derivative of \( f \):

\[
f'(x) = 30e^{\sqrt[3]{3}} \left( \frac{x^{-2/3}}{3} \right) = 10e^{\sqrt[3]{3}}x^{-2/3}
\]

which at the \( x \)-intercept has:

\[
f'\left( \left( \ln(3) \right)^3 \right) = 10e^{\sqrt[3]{3}} \left( \left( \ln(3) \right)^3 \right)^{-2/3} = 10(3)\left( \ln(3) \right)^{-2} = \frac{30}{\left( \ln(3) \right)^2}
\]

Thus, the equation of the tangent line is:

\[
y = f\left( \left( \ln(3) \right)^3 \right) + f'\left( \left( \ln(3) \right)^3 \right) \left( x - \left( \ln(3) \right)^3 \right)
\]

which after plugging in the corresponding values of \( f \) becomes:

\[
y = \frac{30x}{\left( \ln(3) \right)^2} - 30 \ln(3)
\]

6. Determine an equation for each horizontal asymptote on the graph of the following function. Your answer must be justified using limits.

\[
f(x) = \frac{18x^2 - 17x + 16}{2x^2 - e^x}
\]
Solution: \(\lim_{{x \to \infty}} \frac{18x^2 - 17x + 16}{2x^2 - e^x} = 0\) since the dominant term is \(e^x\), in the denominator.
\(\lim_{{x \to -\infty}} \frac{18x^2 - 17x + 16}{2x^2 - e^x} = 9\) since the dominant term is \(x^2\), and we compare coefficients.

Hence \(y = 0\) and \(y = 9\) are the equations of the horizontal asymptotes.

7. Find the derivatives of the following functions. Show all your work:

(a) \(\sin(x^22^x)\)

Solution: \((2x2^x + x^22^x \ln 2) \cos(2^x2^x)\)

(b) \(e^{\cos(x^3)+x^2 \arctan(3\sqrt{x}+7)}\)

Solution:
\[e^{\cos(x^3)+x^2 \arctan(3\sqrt{x}+7)} \left(-3x^2 \sin(x^3) + 2x \arctan(3\sqrt{x} + 7) + \frac{3x^{3/2}}{2 + 2(3\sqrt{x} + 7)^2}\right)\]

(c) \(\sqrt[3]{\ln(3x^2) + 9} + \frac{\sqrt{x^2+1}}{7x^2+3x^6}\)

Solution:
\[
\frac{6x}{3x^2+9} + \frac{x(x^2+1)^{-1/2}(7x^2+3x^6) - \sqrt{x^2+1}(14x+18x^5)}{(7x^2+3x^6)^2}
\]
\[3 \left(\ln(3x^2) + \frac{\sqrt{x^2+1}}{7x^2+3x^6}\right)^{\frac{2}{3}}\]

8. The point \((-1,0)\) is on the curve defined by the equation:

\[x^2y^3 - e^{y^2} = \sqrt[3]{x} + xy\]

Find the value of \(\frac{dy}{dx}\) at \(-1\).

Solution: Perform implicit differentiation to obtain:
\[2xy^3 + x^23y^2 \frac{dy}{dx} - e^{y^2} 2y \frac{dy}{dx} = \frac{1}{3x^{2/3}} + y + x \frac{dy}{dx}\]
Solving this equation for $\frac{dy}{dx}$ yields:

$$\frac{dy}{dx} = \frac{1}{3x^2 y^2 - 2ye^{y^2} - x} \left( \frac{1}{3x^{2/3}} + y - 2xy^3 \right)$$

Plugging $(-1, 0)$ in the right-hand-side yields:

$$\frac{dy}{dx}(-1) = \frac{1}{3(-1)^2(0)^2 - 2(0)e^{0^2} - (-1)} \left( \frac{1}{3(-1)^{2/3}} + 0 - 2(-1)(0)^3 \right) = \frac{1}{3}$$

9. Let $f(x) = 2x^3 - 3x + 7$. Use the definition of a derivative as a limit to prove that $f'(x) = 6x^2 - 3$. Show each step in your calculation and be sure to use proper terminology in each step of your proof.

**Solution:**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)^3 - 3(x+h) + 7 - (2x^3 - 3x + 7)}{h}$$

$$= \lim_{h \to 0} \frac{2(x^3 + 3x^2 h + 3xh^2 + h^3) - 3x - 3h + 7 - 2x^3 + 3x - 7}{h}$$

$$= \lim_{h \to 0} \frac{6x^2 h + 6xh^2 + 2h^3 - 3h}{h}$$

$$= \lim_{h \to 0} \frac{6x^2 h + 6xh^2 + 2h^3 - 3h}{h}$$

$$= \lim_{h \to 0} 6x^2 + 6xh + 2h^2 - 3$$

$$= 6x^2 - 3$$

10. Find $\frac{dy}{dx}$, where $y = \frac{(\ln x)^x}{2^{3x+1}}$. Your final answer must be entirely in terms of $x$.

**Solution:** Applying the natural logarithm, we have

$$\ln y = \ln \left( \frac{(\ln x)^x}{2^{3x+1}} \right)$$

$$= \ln ((\ln x)^x) - \ln (2^{3x+1})$$

$$= x \ln(\ln x) - (3x + 1) \ln 2.$$
Using implicit differentiation, we then obtain
\[
\frac{1}{y} \frac{dy}{dx} = \ln(\ln x) + \frac{x}{x \ln x} - 3 \ln 2.
\]
Finally,
\[
\frac{dy}{dx} = \frac{\ln(x)^x}{2^{x+1}} \left( \ln(\ln x) + \frac{1}{\ln x} - 3 \ln 2 \right).
\]

11. Far off into the future on a mysterious exoplanet, an explorer tosses a marble straight into the air to test the gravity. Her heads-up-display gives her the marble’s height in meters as a function of seconds since it was thrown:
\[h(t) = -3t^2 + 36t + 2\]

As the mission’s intern, you must answer the following questions:
(a) What is the marble’s initial velocity?

**Solution:** The derivative is \(h'(t) = -6t + 36\). Thus, the initial velocity is \(h'(0) = -6(0) + 36 = 36 \text{ m/s}\).

(b) What is the maximum height the marble reaches?

**Solution:** We look for when \(h'(t) = 0\). That is, we solve \(0 = -6t + 36\), which yields \(t = 6\). Thus, the marble reaches the maximum height of \(h(6) = -3(6)^2 + 36(6) + 2 = 110\) meters.

(c) What is the acceleration due to gravity on that planet?

**Solution:** Taking the second derivative we obtain that the acceleration due to gravity is:
\[h''(t) = -6 \text{ m/s}^2\]

12. The curve \(y = f(x)\) has the property that the slope is always equal to three times its \(y\)-coordinate. If the curve passes through the point \((\ln(2), 32)\), give a formula for \(f(x)\).

**Solution:** We are given that \(y' = 3y\), so \(y = Ce^{3x}\). The curves passes through \((\ln(2), 32)\), so \(32 = Ce^{3\ln(2)} = C \cdot 8\), so \(C = 4\). Hence \(f(x) = 4e^{3x}\).