1. (Problem 4.1 from section 7.4 in Durrett)
   
   (a) Generalize the proof of 7.4.6 to conclude that if \( u < v \leq a \) then
   \[
   \mathbb{P}_0(T_a < t, u < B_t < v) = \mathbb{P}_0(2a - v < B_t < 2a - u).
   \]
   
   (b) Let \( M_t = \max_{0 \leq s \leq t} B_s \). Use (a) to derive the joint density
   
   \[
   \mathbb{P}_0(M_t = a, B_t = x) = \frac{2(2a - x)}{\sqrt{2\pi t^3}} e^{-(2a-x)^2/2t}.
   \]

   **Solution:**
   
   (a) Let \( Y_s(w) = 1\{s < t, u < w(t-s) < v\} \) and \( \tilde{Y}_s(w) = 1\{s < t, 2a-v < w(t-s) < 2a-u\} \).
   We have \( \mathbb{E}_a Y_s = \mathbb{E}_a \tilde{Y}_s \). Now use Markov property for the stopping time \( S = \inf\{s < t : B_s = a\} \).
   
   (b) Note that
   
   \[
   \mathbb{P}_0(M_t > a, u < B_t < v) = \mathbb{P}_0(2a - v < B_t < 2a - u).
   \]
   From this we have
   
   \[
   \mathbb{P}_0(M_t > a, B_t = x) = \mathbb{P}_0(B_t = 2a - x) = (2\pi t)^{-1/2} e^{-(2a-x)^2/2t}.
   \]
   Now differentiate w.r.t. \( a \).

2. (Problem 6.2 from section 7.6 in Durrett)
   
   Suppose \( S_n \) is one-dimensional simple random walk and let
   
   \[
   R_n = 1 + \max_{m \leq n} S_m - \min_{m \leq n} S_m
   \]
   
   be the number of points visited by time \( n \). Show that \( R_n/\sqrt{n} \to \) a limit.
   
   **Solution:** \( \phi(\omega) = \max_{0 \leq s \leq 1} \omega(s) - \min_{0 \leq s \leq 1} \omega(s) \) is continuous so we have
   
   \[
   \frac{1}{\sqrt{n}} \left( \max_{m \leq n} S_m - \min_{m \leq n} S_m \right) \to \max_{0 \leq s \leq 1} B_s - \min_{0 \leq s \leq 1} B_s.
   \]

3. (Problem 6.3 from section 7.6 in Durrett)
   
   If \( X_1, X_2, \ldots \) are i.i.d. with \( \mathbb{E}X_i = 0 \) and \( \mathbb{E}X_i^2 = 1 \), then from example 7.6.5 we have
   
   \[
   n^{-3/2} \sum_{m=1}^{n} (n + 1 - m)X_m \to \int_0^1 B_t dt.
   \]
(a) Show that the right hand side has a normal distribution with mean 0 and variance 1/3.

(b) Deduce the result from the Lindeberg-Feller theorem.

**Solution:**

(a) Clearly \( \frac{1}{n} \sum_{m=1}^{n} B(m/n) \) has a normal distribution. The sums converges a.s. and hence in distribution to \( \int_0^1 B_t \, dt \), so by Exercise 3.9 the integral has a normal distribution. To compute the variance, we write

\[
E \left( \int_0^1 B_t \, dt \right)^2 = E \left( \int_0^1 \int_0^1 B_s B_t \, ds \, dt \right) = 2 \int_0^1 \int_s^1 s \, ds \, dt = \frac{1}{3}.
\]

(b) Let \( X_{n,m} = (n + 1 - m)X_m/n^{3/2} \). \( \mathbb{E} X_{n,m} = 0 \) and \( \sum_{m=1}^{n} \mathbb{E} X_{n,m}^2 = n^{-3} \sum_{j=1}^{n} j^2 \rightarrow 1/3 \).

Note that \( \mathbb{E} (X_{n,m}^2; |X_{n,m}| > \varepsilon) \leq n^{-1} \mathbb{E} (X_1^2; |X_1| > \varepsilon \sqrt{n}) \). Hence the sum in (ii) in (4.5) in chapter 2 is \( \leq \mathbb{E} (X_1^2; |X_1| > \varepsilon \sqrt{n}) \rightarrow 0 \) by dominated convergence.

4. (Problem 9.2 from section 7.9 in Durrett)

Show that if \( \mathbb{E} |X_i|^\alpha = \infty \) for some \( \alpha < 2 \) then

\[
\limsup_{n \to \infty} |S_n|/n^{1/\alpha} = \infty \text{ a.s.}
\]

so the law of iterated logarithm fails.

**Hint:** First show that \( \limsup_{n \to \infty} |X_n|/n^{1/\alpha} = \infty \) a.s.

**Solution:** \( \mathbb{E} |X_i|^\alpha = \infty \) implies \( \sum_{m=1}^{\infty} \mathbb{P} (|X_i| > C n^{1/\alpha}) = \infty \) for any \( C > 0 \). Using the second Borel-Cantelli now we see that \( \limsup_{n \to \infty} |X_n|/n^{1/\alpha} \geq C \), i.e., the \( \limsup_{n \to \infty} |X_n|/n^{1/\alpha} = \infty \). Since \( \max\{|S_n|, |S_{n-1}|\} \geq |X_n|/2 \) it follows that \( \limsup S_n/n^{1/\alpha} = \infty \).

5. (Problem 9.3 from section 7.9 in Durrett)

Give a direct proof that the limit set of \( \{S_n/(2n \log \log n)^{1/2}\} \) is \([-1, 1]\), where \( X_1, X_2, \ldots \) are i.i.d. with \( \mathbb{E} X_i = 0, \mathbb{E} X_i^2 = 1 \) and \( S_n = X_1 + \cdots + X_n \).

**Hint:** Use the law of iterated logarithm to get the extreme points and then fill the middle points.

**Solution:** LIL implies that

\[
\limsup_{n \to \infty} S_n/(2n \log \log n)^{1/2} = 1, \liminf_{n \to \infty} S_n/(2n \log \log n)^{1/2} = -1
\]

so the limit set is contained in \([-1, 1]\). On the other hand since \( \mathbb{E} X_i^2 = 1 < \infty \)

\[
\sum_{m=1}^{\infty} \mathbb{P} (X_n > \varepsilon \sqrt{n}) < \infty
\]
for any \( \varepsilon > 0 \). So \( X_n/\sqrt{n} \to 0 \). This shows that the differences \( (S_{n+1} - S_n)/\sqrt{n} \to 0 \). Let \( Y_n = S_n/(2n \log \log n)^{1/2} \). Then we have

\[
\limsup_{n \to \infty} |Y_n - Y_{n-1}| \leq \limsup_{n \to \infty} \frac{|X_n|}{\sqrt{2n \log \log n}} + \limsup_{n \to \infty} |Y_{n-1}| \left| 1 - \sqrt{\frac{2(n - 1) \log \log (n - 1)}{2n \log \log n}} \right| = 0.
\]

So as \( S_n/(2n \log \log n)^{1/2} \) wanders back and forth between \(-1\) and \(1\) it fills up the entire interval.