13.1 Recap and introduction

From Poincare inequality we can get $\text{Var}(f) \leq C \mathbb{E} |f'|^2$. To measure the derivation of a random variable we can use a convex function (say $\Phi$) and define the corresponding concept: $\Phi$-entropy, which we will see in section 13.4 and study the decay of $\Phi$-entropy.

13.2 Self-bounding function

**Definition 13.1** (self-bounding function). $f(x_1, \ldots, x_n)$ is self-bounding function if 
\[ \exists f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \text{ s.t. } 0 \leq f - f_i \leq 1 \text{ and } \sum_{i=1}^{n} (f - f_i) \leq f, \forall i \in \{1, \ldots, n\} \]

**Lemma 13.2.** If $f$ is self-bounding and $x_1, \ldots, x_n$ independent then $\text{Var}(f) \leq \mathbb{E}(f)$

**Proof Sketch:** using Efron-Stein

13.3 Poincare inequality

**Theorem 13.3.** A random vector $x$ satisfy (classical) Poincare inequality with constant $C$ if $\text{Var}(f(\tilde{x})) \leq C \mathbb{E} |\nabla f(\tilde{x})|^2$, $\forall f$ differentiable and $f, f' \in L^2$

**Example 13.4** (Lower bound on tail for a $L$-lipschitz function). Let $f$ be an $L$-lipschitz function, $X = (x_1, \ldots, x_n)$ and $X \sim \text{Poincare}(C)$, $Z = f(x_1, \ldots, x_n)$ and $\bar{Z} = Z - \mathbb{E}Z$ Consider $g(X) = \exp^\theta f(Z)$, since

\[ \text{Var}(g) = \mathbb{E} g^2 - (\mathbb{E} g)^2 = m_{\bar{Z}}(\theta) - (m_{\bar{Z}}(\theta/2))^2 \leq C \]

\[ \sum \left| \frac{\partial}{\partial x_i} g \right|^2 = \sum \left| g \cdot \frac{\theta}{2} \cdot \frac{\partial}{\partial x_i} f \right|^2 = g^2 \cdot \frac{\theta^2}{4} \| \nabla f \|^2 \]

we can get:

\[ \text{Var}(g) \leq C \frac{\theta^2}{4} \mathbb{E} (\| \nabla f \|^2 g^2) \]  

By the $L$-lipschitzness of $f$, we can further get:

\[ m(\theta) - (m_{\theta/2})^2 \leq \left( \frac{CL^2}{4} \right) \theta^2 m(\theta) \]
Take $|\theta| < \frac{2}{L\sqrt{C}} := \alpha$, then
\[
\phi(\theta) = \ln m(\theta) \leq 2\phi\left(\frac{\theta}{2}\right) - \ln \left(1 - \frac{\theta^2}{\alpha^2}\right)
\]
\[
\leq 2^k \phi(\theta/2^k) - \sum_{i=0}^{k-1} \ln \left(1 - \frac{\theta^2}{\alpha^2 2^{2i}}\right) \cdot 2^i
\]
\[
\approx \frac{\theta \phi(\theta/2^k)}{\theta/2^k} - C\theta^2 \alpha^2
\]
And for $\phi$, $\frac{\phi(x)}{x} \to 0$ as $x \to 0$, thus $\ln m_Z \leq \frac{1}{1 - \theta^2/\alpha^2} \frac{2\theta^2}{\alpha^2} \forall |\theta| \leq \alpha = \frac{2}{L\sqrt{C}}$. So,
\[
P(\bar{Z} \geq t) \leq \exp \left(-\frac{t^2}{2(L^2 C + \frac{L\sqrt{C}}{2})}\right), \quad t > 0
\]

Example 13.5. If $Z = (z_1, \ldots, z_n) \sim N(0, I)$, then $Z$ satisfy Poincare(1)

Exercise 13.1. $\sim \Gamma(\alpha, \frac{1}{2})$, then $\text{Var}(f(X)) \leq 2 E(X(f'(X))^2)$

13.4 $\Phi$ - entropy of non-negative random variable

Definition 13.6. $\Phi : [0, \infty) \to [0, \infty)$ is a convex function, then the $\Phi$-entropy of non-negative random variable $X$ is defined as follow: $\text{Ent}_\Phi(X) = E[\Phi(X)] - \Phi(E(X))$
(Not that $\text{Ent}_\Phi(X) \geq 0$ since $\Phi$ is convex.)

Example 13.7. 1) $\Phi(x) = x^2$, then $\text{Ent}_\Phi(X) = \text{Var}(X)$
2) $\Phi(x) = x \log(x)$, then $\text{Ent}_\Phi(X) = \text{Ent}(X) = E[X \log(X)] - E X \log E X$
3) Other examples: $\Phi(x) = x^\alpha$, $1 < \alpha < 2$ or $\Phi(x) = \frac{x^{\alpha-1}}{\alpha-1}$, $\alpha > 1$

Theorem 13.8 (Gaussian $\Phi$-Sobolev Inequality). Let $\Phi : [0, \infty) \to [0, \infty)$ be a convex function and $1/\Phi''$ be concave, for $f : \mathbb{R} \to [0, \infty)$, $\text{Ent}_\Phi(f(z)) \leq \frac{1}{2} E \left[ |f'(z)|^2 \cdot \Phi''(f(z)) \right]$

Proof Sketch: Since
\[
E\Phi(f(z)) - \Phi(E(f(z))) = E\Phi(P_t f(z)) - \Phi(P_\infty f(z))
\]
\[
= -E \int_0^\infty \frac{\partial}{\partial t} \Phi(P_t f(z)) dt
\]
\[
= -E \int_0^\infty \Phi'(P_t f) \cdot A P_t f dt
\]
\[
= E \int_0^\infty \frac{\partial}{\partial x} \Phi'(P_t f) \cdot \frac{\partial}{\partial x} P_t f dt
\]
\[
= E \int_0^\infty \Phi''(P_t f) \cdot e^{-2t} |P_t f'|^2 dt
\]
And we have
\[
P_t f(x) = E[f(OU_t)|OU_0 = x] = E\left[f(e^{-t} x + \sqrt{1 - e^{-2t}} Z)\right], \quad Z \sim N(0, 1)
\]
OU stands for Ornstein-Uhlenbeck semi-group. Also,

$$\left| P_t f \right|^2 = \left| P_t \left( f' \cdot \sqrt{\Phi''(f)} \cdot \frac{1}{\sqrt{\Phi''(f)}} \right) \right|^2$$

$$\leq P_t \left( |f'|^2 \Phi''(f) \right) \cdot P_t \left( \frac{1}{\Phi''(f)} \right)$$

$$\leq P_t \left( |f'|^2 \Phi''(f) \right) \cdot \frac{1}{\Phi''(P_t f)}$$

where in the second line above we use Cauchy-Schwartz inequality and for the third line we use the concavity of $1/\Phi''$. Now we can get:

$$\text{Ent}_{\Phi}(f(z)) \leq \mathbb{E} \int_0^\infty e^{-2t} \cdot P_t \left( |f'|^2 \cdot \Phi''(f) \right) \, dt$$

$$= \frac{1}{2} \mathbb{E} \left( |f'(z)|^2 \cdot \Phi''(f(z)) \right)$$

Example 13.9 (LSI for standard normal random variables). $Z \sim N(0,1)$, then $\mathbb{E} f(Z)(\log f(Z)) - \mathbb{E} f(Z) \cdot \log(\mathbb{E} f(Z)) \leq \frac{1}{2} \mathbb{E} \left[ |f'(Z)|^2 \right]$

Equivalently, $\mathbb{E} f^2(Z)(\log f^2(Z)) - \mathbb{E} f^2(Z) \cdot \log(\mathbb{E} f^2(Z)) \leq 2 \mathbb{E} \left[ |f'(Z)|^2 \right]$

For vector case, if $Z \sim N(0, I_n)$, then $\text{Ent}(f^2(Z)) \leq 2 \mathbb{E} |\nabla f(Z)|^2$

### 13.5 Relationship between Log-Sobolev inequality, Poincare inequality and $\Phi$-entropy

**Claim 13.10.** A random vector $X = (x_1, \ldots, x_n)$ satisfy Log-Sobolev-Inequality with constant $C$ (LSI($C$)) if

$$\text{Ent}(f(X))^2 \leq 2C \cdot \mathbb{E} |\nabla f(X)|^2, \quad \forall f, \nabla f \in L^2$$

**Claim 13.11.** A random vector $X = (x_1, \ldots, x_n)$ satisfy $\Phi$-sobolev if

$$\text{Ent}_{\Phi}(f(X)) \leq C \cdot \mathbb{E} \left[ |\nabla f(X)|^2 \Phi''(f) \right]$$

**Theorem 13.12.** LSI($C$) $\Rightarrow$ Poincare($C$)

**Proof Sketch:** Using Herbst’s argument.