23.1 Martingales

Definition 23.1 (Martingale). Let \( \{M_n\}_{n \geq 0} \) be a stochastic process and \( \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \) is the filtration of \( \{M_n\}_{n \geq 0} \). Then \( \{M_n\}_{n \geq 0} \) is called martingale if each \( M_n \) is \( \mathcal{F}_n \)-measurable, i.e. \( \mathbb{E}|M_n| < \infty \) and \( \mathbb{E}(M_{n+1} | \mathcal{F}_n) = M_n \).

Exercise 23.1. Give an example of a sequence of random variables \( \{X_n\}_{n \geq 0} \) such that

i) \( \{X_n\}_{n \geq 0} \) is a Markov chain but not a martingale;

ii) \( \{X_n\}_{n \geq 0} \) is a martingale but not a Markov chain.

Let \( \epsilon_0, \epsilon_1, \epsilon_2, \ldots \) be a sequence of i.i.d random variables with mean 0. Define \( X_0 = \epsilon_0, X_{n+1} = X_n + \epsilon_{n+1}X_0 \). Then \( \{X_n\}_{n \geq 0} \) is not a Markov chain. On the other hand, one can compute that
\[
\mathbb{E}(X_{n+1} | X_1, \ldots, X_n) = \mathbb{E}(X_n + \epsilon_{n+1}X_0 | X_1, \ldots, X_n) = X_n + X_0 \mathbb{E}\epsilon_{n+1} = X_n.
\]
Therefore, \( \{X_n\}_{n \geq 0} \) is a martingale.

Definition 23.2 (Submartingale). Let \( \{M_n\}_{n \geq 0} \) be a stochastic process and \( \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \) is the filtration of \( \{M_n\}_{n \geq 0} \). Then \( \{M_n\}_{n \geq 0} \) is called submartingale if each \( M_n \) is \( \mathcal{F}_n \)-measurable, i.e. \( \mathbb{E}|M_n| < \infty \) and \( \mathbb{E}(M_{n+1} | \mathcal{F}_n) \geq M_n \).

Definition 23.3 (Supermartingale). Let \( \{M_n\}_{n \geq 0} \) be a stochastic process and \( \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \) is the filtration of \( \{M_n\}_{n \geq 0} \). Then \( \{M_n\}_{n \geq 0} \) is called supermartingale if each \( M_n \) is \( \mathcal{F}_n \)-measurable, i.e. \( \mathbb{E}|M_n| < \infty \) and \( \mathbb{E}(M_{n+1} | \mathcal{F}_n) \leq M_n \).

Example 23.4 (Random walk). Let \( X_0, \ldots, X_n, \ldots \) be a sequence of i.i.d random variables. Let \( S_n = X_1 + \cdots + X_n \). Then \( \{S_n\}_{n \geq 0} \) is a submartingale.

Example 23.5 (Polya’s Urn). Suppose there is an urn that contains red and green balls that are different only by color. At the beginning of the game, the urn only contains 1 red ball and 1 green ball. At each discrete time (trial) \( n \), the player takes out a ball randomly from the urn, and returns the ball along with a new ball of the same color to the urn.

Let \( X_n \) denote the number of red balls at time \( n \) and \( M_n = X_n/n \). Note that
\[
\mathbb{E}(M_{n+1} | M_n = k/n) = \frac{k+1}{n+1} \cdot \frac{k}{n+1} + \frac{k}{n+1} \cdot \left(1 - \frac{k}{n}\right) = \frac{k}{n}.
\]
Therefore, \( \{M_n\}_{n \geq 0} \) is a martingale.

Example 23.6. Given random variables \( X_0, X_1, \ldots, X_n, \ldots \) and \( Y \), let \( M_0 = \mathbb{E}(Y | X_0), \ldots, M_n = \mathbb{E}(Y | X_0, \ldots, X_n) \). Then \( \{M_n\}_{n \geq 0} \) is a martingale. Check that
\[
\mathbb{E}(M_2 | \mathcal{F}_1) = \mathbb{E}(\mathbb{E}(Y | X_0, X_1, X_2) | X_0, X_1) = \mathbb{E}(Y | X_0, X_1).
\]
Example 23.7. Let \( \epsilon_1, \ldots, \epsilon_n, \ldots \) be i.i.d random variables with \( \mathbb{E}(\epsilon_n) = 1 \). Let \( M_n = \prod_{i=1}^{n} \epsilon_i \).

Proposition 23.8. Let \( \{M_n, \mathcal{F}_n\}_{n \geq 0} \) be a martingale. For every \( n \geq 1 \), \( \mathbb{E}M_n = \mathbb{E}M_0 \);

Proposition 23.9. Let \( \{M_n, \mathcal{F}_n\}_{n \geq 0} \) be a martingale and \( T \) be a stopping time, i.e. \( 1\{T = k\} \) is \( \mathcal{F}_k \)-measurable. Then \( \mathbb{E}M_T = \mathbb{E}M_0 \), if one of the following conditions holds,

i) \( T \) is bounded.

ii) \( P(T < \infty) = 1 \) and there is a constant \( C \) such that \( |M_n| \leq C \) for every \( n \leq T \).

Proof. i) Since \( T \) is bounded, there exists a constant \( K \) such that \( T \leq K \). Then

\[
\mathbb{E}(M_T - M_0) = \mathbb{E}\left(\sum_{i=1}^{T-1} (M_{i+1} - M_i)\right) = \mathbb{E}\left(\sum_{i=1}^{K-1} (M_{i+1} - M_i)1\{i \leq T-1\}\right)
\]

\[
= \sum_{i=1}^{K-1} (\mathbb{E}(M_{i+1} - M_i)1\{i \leq T-1\}) = \sum_{i=1}^{K-1} 0 = 0.
\]

where the second last equality holds since \( 1\{i \leq T-1\} \) is \( \mathcal{F}_T \)-measurable and \( \mathbb{E}M_{i+1} = \mathbb{E}M_i \).

ii) For every \( m \in \mathbb{N} \), let \( S = T \wedge m \). Then \( S \) is also a stopping time. From part i), we have \( \mathbb{E}M_S = \mathbb{E}M_0 \). Therefore,

\[
|\mathbb{E}M_T - \mathbb{E}M_0| = |\mathbb{E}M_T - \mathbb{E}M_{T \wedge m}| \leq \mathbb{E}|M_T - M_{T \wedge m}| \leq 2C \cdot \mathbb{P}(T > m) \rightarrow 0, \text{ as } m \rightarrow \infty.
\]

Example 23.10 (Simple symmetric random walk). Let \( \{\epsilon_n\}_{n \geq 0} \) be a sequence of i.i.d random variable with distribution \( (P)(\epsilon_n = -1) = (P)(\epsilon_n = 1) = \frac{1}{2} \). Let \( S_0 = \epsilon_n \) and \( S_{n+1} = S_n + \epsilon_{n+1} \). Then \( \{S_n\}_{n \geq 0} \) is a simple symmetric random walk. Let \( T = \inf\{n \mid S_n = a \text{ or } S_n = b\} \).

1. Find \( \mathbb{P}(S_T = a) \).

Let \( p = \mathbb{P}(S_T = a) \). Then \( \mathbb{E}(S_T) = a \cdot \mathbb{P}(S_T = a) + b \cdot \mathbb{P}(S_T = b) = ap + b(1 - p) \). On the other hand, since \( S_T \) satisfies the condition ii) on Proposition 23.9, we have \( \mathbb{E}(S_T) = \mathbb{E}(S_0) = 0 \). Therefore, we obtain \( ap + b(1 - p) = 0 \), which gives \( p = \frac{a}{b - a} \).

2. Find \( \mathbb{E}(T) \).

Let \( M_n = S_n^2 - n \). First, we will show \( \{M_n\}_{n \geq 0} \) is a martingale. Note that

\[
\mathbb{E}(S_{n+1}^2 \mid S_n) = \mathbb{E}((S_n + \epsilon_n)^2 \mid S_n) = \mathbb{E}(S_n^2 + 2S_n\epsilon_n + \epsilon_n^2 \mid S_n) = S_n^2 + 1.
\]

Therefore,

\[
\mathbb{E}(M_{n+1} \mid M_n) = \mathbb{E}(S_{n+1}^2 - (n + 1) \mid M_n) = S_n^2 + 1 - (n + 1) = S_n^2 - n.
\]

By Proposition 23.9, \( \mathbb{E}M_T = \mathbb{E}M_0, \) i.e. \( \mathbb{E}(S_T^2 - T) = \mathbb{E}(S_0^2 - 0) = 0 \). Therefore, we get

\[
\mathbb{E}T = \mathbb{E}S_T^2 = a^2 \cdot \frac{b}{b - a} + b^2 \cdot \frac{-a}{b - a} = ab.
\]
Exercise 23.2. Let $S_n$ be a biased random walk.

- Show that $M_n = (\frac{1-p}{p})^n S_n$ is a martingale.
- Calculate $\mathbb{P}(M_n \text{ hits } a \text{ before } b)$. 