22.1 Class Structure

Consider a rate matrix $Q$ on the state space $I$, where $I$ is countable. Consider a CTMC $(X_t)_{t\geq 0} \sim Markov(\lambda, Q)$, where $X_0 \sim \lambda$, $Q$ is the rate matrix, $(Y_n)_{n\geq 0}$ is the jump chain and $J_n = S_1 + \ldots + S_n, n \geq 1$ are the jump times. The transition matrix for $X(t)$ is $P(t) = (p_{ij}(t))_{i,j \in I}$, where $p_{ij}(t) := \mathbb{P}(X(t) = j \mid X(0) = i)$.

**Definition 22.1.** We say that $i$ leads to $j$ and write “$i \rightarrow j$” if

$$\mathbb{P}_i(X(t) = j \text{ for some } t \geq 0) > 0.$$ and $i \leftrightarrow j$ iff both $i \rightarrow j$ and $j \rightarrow i$.

**Corollary 22.2.** I can be decomposed into disjoint communicating classes.

**Theorem 22.3.** For distinct states $i$ and $j$ the following are equivalent:

(i) $i \rightarrow j$;

(ii) $i \rightarrow j$ for the jump chain;

(iii) $q_{i_{1}}, q_{i_{1}i_{2}} \ldots q_{i_{k}j} > 0$ for some states $i_1, i_2, \ldots, i_k$;

(iv) $p_{ij}(t) > 0$ for all $t > 0$;

(v) $p_{ij}(t) > 0$ for some $t > 0$.

**Proof.** We will prove that iii) implies iv). First we claim that $q_{ij} > 0$ implies that $p_{ij}(t) > 0$ for all $t > 0$. We have

$$p_{ij}(t) = \mathbb{P}_i(X_t = j) \geq \mathbb{P}_i(S_1 < t, Y_1 = j, S_2 > t) = \mathbb{P}_i(T_1 < t, Y_1 = j, T_2 > t) = (1 - e^{-q_{ij}t}) \cdot \frac{q_{ij}}{q_{i}} \cdot e^{-q_{j}t} > 0.$$

Thus,

$$P_{ij}(t) \geq P_{i_{1}i_{1}}(\frac{t}{k+1})P_{i_{1}i_{2}}(\frac{t}{k+1}) \ldots P_{i_{k}j}(\frac{t}{k+1}) > 0.$$ ■

**Corollary 22.4.** If the jump chain is irreducible,

$$p_{ij}(t) > 0 \quad \forall i, j, t > 0.$$

i.e., the DTMC $(Z_n = X_{nh})_{n \geq 0}$ for some $h > 0$ fixed, is aperiodic and irreducible.
22.2 Hitting Times, Absorption Time and Return Times

22.2.1 Hitting Times

Given a set \( A \subseteq I \), the Hitting Time of \( A \) is defined as

\[
D^A := \inf\{t \geq 0 | X_t \in A\}
\]

If \( A \) is closed, \( D^A \) is also called Absorption Time.

22.2.2 Return Times

\[
T^A := \inf\{t \geq J_1 | X_t \in A\}
\]

Remark 22.5. If \( Y_0 = i, T_i \geq \text{dist}(i, A) \) (for DTMC). But for CTMC, \( T_i \) can be any non-negative number.

Define

\[
h_i^A = P_i(D^A < \infty), \quad k_i^A = E_i(D^A)
\]

Claim 22.6.

\[
D^A = J_{H^A}, \text{ if } H^A < \infty,
\]

\[
D^A = \infty, \text{ o.w.}
\]

Proof. Use \( X_t = Y_n \), if \( J_n \leq t < J_{n+1} \). If \( H^A \leq \infty \),

\[
\begin{cases}
\zeta < \infty \Rightarrow D^A = \inf \emptyset = \infty \\
\zeta = \infty \Rightarrow J_n \uparrow \infty, H^A = \infty \Rightarrow D^A = \infty
\end{cases}
\]

Theorem 22.7. Both \( h^A := (h_i^A : i \in I) \) and \( k^A := (k_i^A : i \in I) \) are minimal non-negative solutions to the following equations

\[
\begin{cases}
h_i^A = 1, \text{ if } i \in A \\
\sum_{j \in I} q_{ij} h_j^A = 0, \text{ if } i \notin A
\end{cases}
\]

\[
\begin{cases}
k_i^A = 0, \text{ if } i \in A \\
\sum_{j \in I} q_{ij} k_j^A = -1, \text{ if } i \notin A
\end{cases}
\]
### 22.3 Recurrence and Transience

**Definition 22.8.** We say a state $i$ is recurrent if

$$P_i(\{t \geq 0 : X_t = i\} \text{ is unbounded}) = 1.$$  

We say that $i$ is transient if

$$P_i(\{t \geq 0 : X_t = i\} \text{ is unbounded}) = 0.$$  

**Theorem 22.9.** We have:

(i) if $i$ is recurrent for the jump chain $(Y_n)_{n \geq 0}$, then $i$ is recurrent for CTMC $(X_t)_{t \geq 0}$;

(ii) if $i$ is transient for the jump chain $(Y_n)_{n \geq 0}$, then $i$ is transient for CTMC $(X_t)_{t \geq 0}$;

(iii) every state is either recurrent or transient;

(iv) recurrence and transience are class properties.

**Proof.**

(i) Suppose $i$ is recurrent for the jump chain $(Y_n)_{n \geq 0}$. If $X_0 = i$ then $(X_t)_{t \geq 0}$ does not explode and $J_n \uparrow \infty$ w.p. 1, but $T_1, T_2, \ldots$ are the return times to $i$ in the jump chain and $T_i \uparrow \infty$.

$$\{t \geq 0 | X_t = i\} \supseteq \{J_{T_i}, i \geq 1\} \implies \{t \geq 0 | X_t \leq i\} \text{ is unbounded w.p. 1.}$$

(ii) Suppose $i$ is transient for the jump chain $(Y_n)_{n \geq 0}$. If $X_0 = i$ then

$$\exists N < 0, \text{ s.t. } Y_n \neq i, \forall n \geq N$$

$$\implies X_t \neq i, \forall t \geq J_N$$

$$\implies \sup |\{t \geq 0 | X_t = i\}| \leq J_n < \infty$$

$$\implies P_i(\{t \geq 0 : X_t = i\} \text{ is unbounded}) = 0.$$  

**Theorem 22.10** (Dichotomy). The following dichotomy holds:

(i) if $q_i = 0$ or $i$ is recurrent for the jump chain, then

$$i \text{ is recurrent for CTMC and } \int_0^\infty P_i(t)dt = \infty.$$  

(ii) if $q_i > 0$ or $i$ is transient for the jump chain, then

$$i \text{ is transient for CTMC and } \int_0^\infty P_i(t)dt < \infty.$$  

**Proof.** Write $\pi_{ij}^{(n)}$ for the $(i,j)$ entry in $\Pi^n$. We shall show that when $q_i > 0$, \ldots
\[
\int_0^\infty P_{ii}(t)\,dt = \frac{1}{q_i} \sum_{n=0}^\infty \pi_{ii}^{(n)}.
\]

LHS = \int_0^\infty P_i(X_t = i)\,dt = \int_0^\infty \mathbb{E}_i(1_{\{X_t = i\}})\,dt = \mathbb{E}_i[\int_0^\infty 1_{\{X_t = i\}}\,dt]

= \mathbb{E}_i\left(\sum_{n=0}^\infty 1_{\{Y_n = i\}}\right) = \sum_{n=0}^\infty \frac{1}{q_i} P(Y_n = i) = \frac{1}{q_i} \sum_{n=0}^\infty |p_{ii}^{(n)}| = \text{RHS}.

\[\blacksquare\]

### 22.4 Invariant Measure for CTMC

**Definition 22.11.** A measure \( \lambda \) is invariant for \( Q \) if

\( \lambda Q = 0 \).

**Remark 22.12.** \( P(t) = e^{tQ} \).

If \( \lambda Q = 0 \implies \lambda P(t) = \lambda e^{tQ} = \lambda(I + tQ + \frac{tQ^2}{2!} + \ldots) = \lambda + 0 = \lambda \).

**Theorem 22.13.** \( \lambda Q = 0 \) iff \( \lambda q \Pi = \lambda q \), where \( (\lambda q)_i = \lambda_i q_i \).

**Proof.** \( \forall i \),

\[\sum_{j \in I} \lambda_j q_{ji} = 0 \implies -\lambda_i q_i + \sum_j \lambda_j q_j \pi_j = 0 \implies \sum_j (\lambda q)_j \pi_{ij} = (\lambda q)_i.\]

\[\blacksquare\]

**Lemma 22.14.** \( nh \leq i \leq (n+1)h \) for \( h > 0 \),

\[e^{-hq_j} P(nh) \leq P_{ij}(t) \leq e^{-hq_j} P_{ij}((n+1)h).\]

**Proof.**

\[P_{ij}(t) = P_i(X_t = j) \geq P_i(X_{nh} = j, \text{holding time} \geq h \text{ after } nh) = P_i(X_{nh} = j) = e^{-q_i h}.\]

\[\blacksquare\]