Poisson processes are some of the simplest examples of continuous-time Markov chains. We shall also see that they may serve as building blocks for the most general continuous-time Markov chain. Moreover, a Poisson process is the natural probabilistic model for any uncoordinated stream of discrete events in continuous time. So we shall study Poisson processes first, both as a gentle warm-up for the general theory and because they are useful in themselves. The key result is Theorem 2.4.3, which provides three different descriptions of a Poisson process. The reader might well begin with the statement of this result and then see how it is used in the theorems and examples that follow. We shall begin with a definition in terms of jump chain and holding times.

**Definition 19.1.** Define a right-continuous process \((X_t)_{t \geq 0}\) with values in \(\{0, 1, 2, \ldots\}\) and holding times \(S_1, S_2, \ldots\) are independent exponential random variables of parameter \(\lambda\) and its jump chain is given by \(\gamma_n = n\). Here is the diagram:

\[
Q = \begin{pmatrix}
-\lambda & 0 & 0 & \cdots & \\
0 & -\lambda & 0 & 0 & \cdots \\
0 & 0 & -\lambda & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots 
\end{pmatrix}
\]

**Theorem 19.2.** (Markov property). Let \((X_t)_{t \geq 0}\) be a Poisson process of rate \(\lambda\). Then, for any \(s \geq 0\), \((X_{s+t} - X_s)_{t \geq 0}\) is also a Poisson process of rate \(\lambda\) independent of \((x_r : r \leq s)\).

**Proof.** It suffices to prove the claim conditional on the event \(X_s = i\), for each \(i \geq 0\). Set \(\hat{X}_t = X_{s+t} - X_s\), We have \(\{X_s = i\} = \{J_i \leq s < J_i + 1\} = \{J_i \leq s\} \cap \{S_i + 1 > s - J_i\}\) on this event \(X_r = \sum_{j=1}^{i} 1_{\{S_j \leq r\}}\) for \(r \leq s\) and the holding times \(\hat{S}_1, \hat{S}_2, \ldots\) of \((\hat{X}_t)_{t \geq 0}\) are given by \(\hat{S}_1 = S_i + 1 - (s - J_i), \hat{S}_n = S_i + n\) for \(n \geq 2\).

**Definition 19.3.** (Strong Markov property). Let \((X_t)_{t \geq 0}\) be a Poisson process of rate \(\lambda\) and let \(X_T\) be a stopping time of \((X_t)_{t \geq 0}\). Then, conditional on \(T < \infty\), \((X_{T+t} - X_T)_{f \geq 0}\) is also a Poisson process of rate \(\lambda\) independent of \((x_s : s \leq T)\).

Here is some standard terminology. If \((X_t)_{t \geq 0}\) is a real-valued process, we consider its increment \(X_t - X_s\) over any interval \([s, t]\). We say that \((X_t)_{t \geq 0}\) has stationary increments if the distribution of \(X_{s+t} - X_s\) depends only on \(t \geq 0\). We say that \((X_t)_{t \geq 0}\) has independent increments if its increments over any finite collection of disjoint intervals are independent.

**Theorem 19.4.** Let \((X_t)_{t \geq 0}\) be an increasing, right continuous integer valued process starting from 0. Let \(0 < \lambda < \infty\). Then the following three conditions are equivalent:

(a) (jump chain/holding time definition) the holding times \(S_1, S_2, \ldots\) of \((X_t)_{t \geq 0}\) are independent exponential random variables of parameter \(\lambda\) and the jump chain is given by \(Y_T = n\) for all\(n\);
(b) (infinitesimal definition) \( (X_t)_{t \geq 0} \) has independent increments and, as \( h \downarrow 0 \), uniformly in \( t \),
\[ \mathbb{P}(X_t + h - X_t = 0) = 1 - \lambda h + o(h), \mathbb{P}(X_t + h - X_t = 1) = \lambda h + o(h) \]

(c) (transition probability definition) \( (X_t)_{t \geq 0} \) has stationary independent increments and, for each \( t \), \( X_t \) has Poisson distribution of parameter \( \lambda t \).

If \( (X_t) \rightarrow 0 \) satisfies any of these conditions then it is called a Poisson process of rate \( \lambda \).

Proof. (a) \( \Rightarrow \) (b) If (a) holds, then, by the Markov property, for any \( t, h \geq 0 \), the increment \( X_{t+h} - X_t \) has the same distribution as \( X_h \) and is independent of \( (X_s : s \leq t) \). So \( (X_t)_{t \geq 0} \) has independent increments and as \( h \uparrow 0 \)
\[ \mathbb{P}(X_t + h - X_t \geq 1) = \mathbb{P}(X_h \geq 1) = \mathbb{P}(J_1 \leq h) = 1 - e^{-\lambda h} = \lambda h + o(h) \]
\[ \mathbb{P}(X_t + h - X_t \geq 2) = \mathbb{P}(X_h \geq 2) = \mathbb{P}(J_2 \leq h) \leq \mathbb{P}(S_1 \geq h, S_2 \leq h) = (1 - e^{-\lambda h})^2 = o(h) \]
which implies (b).

(b) \( \Rightarrow \) (c) If (b) holds, then, for \( i = 2, 3, \ldots \), we have \( \mathbb{P}(X_t + h - X_t = i) = o(h) \) as \( h \downarrow 0 \), uniformly in \( t \). Set \( p_j(t) = \mathbb{P}(X_t = j) \)
Then, for \( j = 1, 2, \ldots \)
\[ p_j(t + h) = \mathbb{P}(X_t + h = j) \]
\[ = \sum_{i=1}^{j} \mathbb{P}(X_t + h - X_t = i) \mathbb{P}(X_t = j - i) \]
\[ = (1 - \lambda h + o(h))p_j(t) + (\lambda h + o(h))p_j - 1(t) + o(h) \]
so
\[ \frac{p_j(t+h) - p_j(t)}{h} = -\lambda p_j(t) + \lambda p_{j-1}(t) + o(h) \]
Since this estimate is uniform in \( t \) we can put \( t = s - h \) to obtain for all \( s \geq h \)
\[ \frac{p_j(s) - p_j(s-h)}{h} = -\lambda p_j(s-h) + \lambda p_{s-h}(t) + o(h) \]
Now let \( h \downarrow 0 \) to see that \( p_j(t) \) is first continuous and then differentiable and satisfies the differential equation
\[ p_j'(t) = -\lambda p_j(t) + \lambda p_{j-1}(t) \]
By a simpler argument we also find
\[ p_j'(t) = -\lambda p_0(t) \]
Since \( X_0 = 0 \) we have initial conditions
\[ p_0(0) = 1, p_j(0) = 0 \] for \( j = 1, 2, \ldots \)

This system of equations has a unique solution given by
\[ p_j(t) = e^{-\lambda t} \frac{(\lambda t)^j}{j!}, j = 0, 1, 2, \ldots \]
Hence \( X_t \sim \mathcal{P}(\lambda t) \). If (X) satisfies (b), then certainly \( (X_t)_{t \geq 0} \) has independent increments, but also \( (X_{s+t} - X_s)_{t \geq 0} \) satisfies (b), so the above argument shows \( X_{s+t} - X_s \sim \mathcal{P}(\lambda t) \), for any \( s \), which implies (c).

(c) \( \Rightarrow \) (a) There is a process satisfying (a) and we have shown that it must then satisfy (c). But condition (c) determines the finite-dimensional distributions of \( (X_t)_{t \geq 0} \) and hence the distribution
of jump chain and holding times. So if one process satisfying (c) also satisfies (a), so must every process satisfying (c). □