7.0.1 Strong Markov Property

**Definition 7.1.** Consider a Markov chain \((X_n)\) and the associated filtration \(\mathcal{F}_m\). A random variable \(T: \Omega \to \mathbb{N} \cup \{\infty\}\) is called a stopping time if the event \(\{T = m\} \in \mathcal{F}_m \\forall m \geq 0\), or \(\{T \leq m\} \in \mathcal{F}_m \\forall m \geq 0\).

For a given \(\sigma\)-field \((\Omega, \mathcal{F}, \mathbb{P})\) and filtration \(\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \ldots\)

Let define, \(\mathcal{F}_T = \{A \in \mathcal{F} | A \cap \{T = m\} \in \mathcal{F}_m\} \\forall m \geq 0\)

Claim: \(\mathcal{F}_T\) is a \(\sigma\)-field, \(\mathcal{F}_T\) is the information contained in \(x_0, x_1, x_2, \ldots x_T\) up to time \(T\)

**Theorem 7.2.** (Strong Markov Property) Let \((X_i)_{i \geq 0} \sim \text{Markov}(\lambda, \mathbb{P})\) be discrete markov chains and \(T\) be stopping time of \((X_n)\). Then condition on \(T < \infty\) and \(X_T = i\) the stochastic process \((X_{T+j})_{j \geq 0} \sim \text{Markov}(\delta_i, \mathbb{P})\) and is independent of \(\mathcal{F}_T\)

**Proof.** Let \(A \in \mathcal{F}_T\) \(A\) is any event that can be determined by \(X_1, X_2, \ldots X_T\). By definition, \(A \cap \{T = m\} \in \mathcal{F}_m\), thereby using the standard Markov Property at time \(m\), we have

\[
\mathbb{P}(\{X_{T+1} = X_1 \ldots X_{T+n} = X_n\} \cap A \cap \{T < \infty, X_T = i\}) = \mathbb{P}(\{X_1 = X_1 \ldots X_n = X_n\}) \mathbb{P}(A \cap \{T < \infty, X_T = i\})
\]

It’s enough to prove

\[
\mathbb{P}(\{X_{T+1} = X_1 \ldots X_{T+n} = X_n\} \cap A \cap \{T = m, X_T = i\}) = \mathbb{P}(\{X_1 = X_1 \ldots X_n = X_n\}) \mathbb{P}(A \cap \{T = m, X_T = i\}), \ \forall m \geq 0
\]

We know that \((A \cap \{T = m\}) \in \mathcal{F}_m\)

\[
\mathbb{P}(\{X_{T+1} = X_1 \ldots X_{T+n} = X_n\} \cap A \cap \{T = m\} \cap \{X_T = i\}) = \mathbb{P}(\{X_{T+1} = X_1 \ldots X_{T+n} = X_n\} \cap (A \cap \{T = m\}) \cap \{X_T = i\})
\]

From definition of Markov chain, future is independent of past

\[
= \mathbb{P}_i(X_1 = X_1, \ldots, X_n = X_n) \mathbb{P}(A \cap \{T = m\} \cap \{X_T = i\})
\]
Example 1: Given a $\sigma$-field $(\Omega, \mathcal{F}, \mathbb{P})$, $X_0, X_1, X_2, \ldots \sim \text{Markov}(\lambda, P)$. With almost sure finite hitting time $\mathbb{P}(H_A < \infty) = 1$, \forall i

\[
T_0 = H_A \quad \text{(First time when chain hits A)} \\
T_1 = \inf\{n \geq T_0 + 1 \mid x_n \in A\} \quad \text{(Second hitting time)} \\
\vdots \\
T_m = \inf\{n \geq T_{m-1} + 1 \mid x_n \in A\} \quad \text{(m hitting time)}
\]

Claim 7.3. $(X_{T_j})_{j \geq 0}$ is a markov chain with state space $A$

Proof. Let $A \subset E$, $(E$ being the state space),

\[
\mathbb{P}(X_{T_0} = j) = \mathbb{P}(X_{T_0} = j \mid X_0 = \lambda), j \in A \]

\[
= \mathbb{P}(X_{T_0+1} = i_1, \ldots X_{T_0+n} = i_n \mid X_{T_0} = j)
\]

\[
= \mathbb{P}(X_{T_{\tilde{1}}} = j) \quad \text{(Using Strong Markov Property)}
\]

This Implies

\[
\mathbb{P}(X_{T_1} = j_1, \ldots X_{T_m} \mid X_{T_0} = j) = \mathbb{P}(X_{T_{\tilde{1}}} = j_1, \ldots X_{T_{\tilde{m}}} = j_m)
\]

where

\[
\tilde{T}_1 = \inf\{i \geq 1 \mid X_i \in A\} \\
\vdots \\
\tilde{T}_m = \inf\{i \geq \tilde{T}_{m+1} + 1 \mid X_i \in A\}
\]

The Transition probability can be written as

\[
\tilde{P} = \mathbb{P}(X_{T_1} = j \mid X_{T_0} = i) \\
= \mathbb{P}_i(X_{\tilde{T}_1} = j), \quad \text{where } \tilde{T}_1 : \text{First return time to A}
\]

Theorem 7.4. $T_m$'s are stopping time \forall \; m \geq 0

Proof. \quad \bullet \; \text{Step 1: Prove } T_0 \text{ is stopping time}

\bullet \; \text{Step 2: Use induction to show that if } T_{m-1} \text{ is a stopping time, then } T_m \text{ is stopping time.}

\{T_m = k\} \in \mathcal{F}_k, \forall k
Example 7.5. Random walk

Let $A = \{0\}$, then

$$H_A = \mathbb{P}_k(H_A < \infty) = \begin{cases} 1 - \left(\frac{q}{p}\right)^k, & \text{if } p \neq q \\ 1 - \frac{k}{N}, & \text{otherwise} \end{cases}$$

$$= \mathbb{P}(\text{hit 0 before hitting N})$$

As $N \to \infty$

$$\mathbb{P}_k(\text{Hit 0}) = \begin{cases} 1, & \text{if } p \leq q \\ \left(\frac{q}{p}\right)^k, & \text{if } p \geq q \end{cases}$$

7.0.2 Probability Generating function for stopping time

Definition 7.6. Let $Y$ be integer valued random variable, taking non-negative values, i.e. $Y \in \{0, 1, 2, \ldots\}$ and the probability $Y = k$ is defined as

$$p_k = \mathbb{P}(Y = k)$$

then the probability generating function of $Y$ is defined as

$$\Phi_Y(t) = \mathbb{E}(t^Y) \quad t \in [0, 1)$$

$$= \sum_{k=0}^{\infty} t^k \mathbb{P}(Y = k)$$

Claim 7.7. The probability generating function $\Phi_y$ is always upper bounded by 1

$$\Phi_y : [0, 1) \to [0, 1]$$

Example 7.8. Random walk
Defining hitting time $H_0$ as
\[ H_0 = \inf\{n \geq 0 \mid X_n = 0\} \]
Then
\[ \mathbb{P}_k(H_A < \infty) = \begin{cases} 1 & \text{if } q \geq p \\ \left(\frac{q}{p}\right)^k & \text{if } q \leq p \end{cases} \]
For simplicity let's call $H = H_{\{0\}}$ and for our example

\[ \Phi(t) = \mathbb{E}_1(t^{H_0}), \quad 0 \leq t \leq 1 \]
\[ \Phi(t) = q \mathbb{E}(t^{H_0} \mid X_0 = 1) + p \mathbb{E}_1(t^{H_0} \mid X_1 = 2, X_2 = 1) \]  
(7.1)
From the above graph we can write the hitting time for starting in node 2 as
\[ H_0 = 1 + (H_1 \mid \text{start from } 2) + (\tilde{H}_0 \mid \text{start from } 1) \]
where $\tilde{H}_0$ is copy of $H_0$.
Therefore, for condition $H_1$ is finite i.e. $(H_1 < \infty)$
\[ \mathbb{E}(t^{H_0} \mid X_1 = 2, X_2 = 1) = t \mathbb{E}(t^{H_1} \times t^{\tilde{H}_0}) \]
(7.2)
\[ = t \mathbb{E}(t^{H_1} \times t^{\tilde{H}_0}) \]
(7.3)
From strong Markov Property
\[ t \mathbb{E}(t^{H_1} \times t^{\tilde{H}_0}) = t \mathbb{P}_2(H_1 < \infty)\mathbb{E}_2(t^{H_1} \mid H_1 < \infty)\mathbb{E}_1(t^{\tilde{H}_0}) \]
(7.4)
\[ = t \mathbb{E}_2(t^{H_1}, 1_{H_1 < \infty})\mathbb{E}_1(t^{\tilde{H}_0}) \]
(7.5)
\[ = t \mathbb{E}_2(t^{H_1})\mathbb{E}_1(t^{\tilde{H}_0}) \]
(7.6)
\[ = t \Phi(t) \Phi(t) \]
(7.7)
We can take out the term $1_{H_1 < \infty}$ because of the fact for $t \in [0, 1), t^\infty = 0$
\[ t \mathbb{E}_2(t^{H_1})\mathbb{E}_1(t^{\tilde{H}_0}) = t \Phi(t) \Phi(t) \]
(7.8)
Therefore by combining the terms we can write
\[ \Phi(t) = qt + pt \Phi(t)^2 \]
(7.9)
Solving for $\Phi(t)$, finding the roots of the quadratic equations

$$\Phi(t) = \frac{1 - \sqrt{1 - 4pq^2}}{2pt} \quad \text{or} \quad \Phi(t) = 1, \forall t \in [0, 1) \quad (7.10)$$

Taking limit as $t \to 1$

$$\lim_{t \uparrow 1^{-}} \Phi(t) = \lim_{t \uparrow 1^{-}} \mathbb{E}_1(t^{H_0}) \quad (7.11)$$

$$\quad = \lim_{t \uparrow 1^{-}} \mathbb{E}_1(1.1_{H_0<\infty}) \quad (7.12)$$

$$\quad = \mathbb{P}(H_0 < \infty) \quad (7.13)$$

Limit of RHS

$$\lim_{t \uparrow 1^{-}} \Phi(t) = \frac{1 - \sqrt{1 - 4p(1 - p)}}{2p} \quad (7.14)$$

$$\quad = \frac{1 - \sqrt{(1 - 2p)^2}}{2p} \quad (7.15)$$

$$\quad = \frac{1 - |1 - 2p|}{2p} \quad (7.16)$$

$$\mathbb{P}(H_0 < \infty \mid X_0 = 1) = \begin{cases} 1, & \text{if } p \leq q \\ \left(\frac{q}{p}\right), & \text{if } p \geq q \end{cases}$$

$(H_0 < \infty \mid X_0 = k)$ is sum of $k$ many i.i.d copies of $(H_0 < \infty \mid X_0 = 1)$, therefore the term $\mathbb{P}(H_0 < \infty \mid X_0 = k)$ can be written as

$$\mathbb{P}(H_0 < \infty \mid X_0 = k) = \begin{cases} 1, & \text{if } p \leq q \\ \left(\frac{q}{p}\right)^k, & \text{if } p \geq q \end{cases}$$