Final Exam

MATH 564/STAT 555: Applied Stochastic Processes

Due date: December 19, 2017

Answer all questions. The maximum point one can get is 60.

1. (10 pts) Let $\xi_i, i \geq 1$ be i.i.d. r.v.’s with $P(\xi_i = +1) = p$ and $P(\xi_i = -1) = q := 1 - p$ with $p \neq q$. Put $X_0 = 0$ and $X_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \geq 1$. For a positive integer $a$ define the stopping time

$$T := \inf\{n \geq 0 \mid |X_n| = a\}.$$

Find $E(T)$.

**Hint:** Use appropriate martingale.

**Solution:** Consider the two processes

$$M_n := (q/p)^{X_n}, \quad N_n := X_n - (p - q)n \quad \text{for } n \geq 0.$$

We claim that both $(M_n)_{n \geq 0}$ and $(N_n)_{n \geq 0}$ are martingales w.r.t the filtration $\mathcal{F}_n = \sigma(X_0, X_1, X_2, \ldots, X_n), n \geq 0$. It is easy to see that both $M_n, N_n$ are $\mathcal{F}_n$ measurable for $n \geq 0$. Moreover, $|M_n| \leq \max\{(q/p)^n, (p/q)^n\}, |N_n| \leq 2n$ for all $n$. The martingale property follows from the fact that $E((q/p)^{\xi_n}) = 1$ and $E(\xi_n - (p - q)) = 0$.

Now we can use Optional Stopping Theorem for the Martingale $M_n, n \geq 0$ with the stopping time $T$ which is finite a.s. and $|M_n| \leq \max\{(q/p)^a, (p/q)^a\}$ for $n \leq T$, to get

$$1 = E(M_T) = (q/p)^a P(X_T = a) + (p/q)^a P(X_T = -a) = (q/p)^a P(X_T = a) + (p/q)^a (1 - P(X_T = a))$$

or

$$P(X_T = a) = \frac{1 - (p/q)^a}{(q/p)^a - (p/q)^a} = \frac{p^a}{p^a + q^a}, \quad P(X_T = -a) = \frac{q^a}{p^a + q^a}.$$

Similarly, applying OST for the bounded stopping time $T \land m$ for the martingale $N_n$ we get

$$E(N_{T \land m}) = E(N_0) = 0 \quad \text{or} \quad E(T \land m) = E(X_{T \land m}) = \frac{E(X_T)}{p - q} \leq \frac{a}{p - q}.$$

Taking $m \uparrow \infty$ and using Monotone convergence theorem and Dominated convergence theorem, we get

$$E(T) = \frac{E(X_T)}{p - q} = \frac{a(p^a - q^a)}{(p - q)(p^a + q^a)}.$$

2. (4+5+3+3 pts) Suppose that customers arrive for service according to a Poisson process with parameter $\lambda$ and each customer starts being served immediately upon his arrival (i.e., there are infinite number of servers). Suppose that service times are independent and exponentially distributed with parameter $\mu$. Let $X(t), t \geq 0$, be the number of customers in the process of being served at times $t$. Clearly, $X = (X(t))_{t \geq 0}$ is a birth and death process ($\text{M/M}/\infty$ process).

(a) Let $X_1(t)$ be the number of customers arriving in $(0, t]$ that are still in the process of being served at time $t$. Show that $X_1(t)$ is a Poisson random variable with parameter

$$\frac{\lambda}{\mu}(1 - e^{-\mu t}).$$

**Hint:** You can use the fact if $Y(t)$ denotes the number of customers arriving in $(0, t]$, then $(Y(t))_{t \geq 0}$ is a Poisson process. And then you can use properties of Poisson processes.
(b) Let \( i \) be the number of customers present initially and let \( X_2(t) \) be the number of these customers still in the process of being served at time \( t \). Obviously \( X_2(t) \) is independent of \( X_1(t) \) and has a binomial distribution with parameter \( i \) and \( e^{-\mu t} \). Use these facts to find
\[
p_{ij}(t) = P(X(t) = j | X(0) = i).
\]

(c) Suppose that the initial distribution \( \pi_0 \) is a Poisson distribution with parameter \( \nu \). Show that \( X_2(t) \) has a Poisson distribution with parameter \( \nu e^{-\mu t} \).

(d) Show that if the initial distribution \( \pi_0 \) is a Poisson distribution with parameter \( \nu \), then \( X_t \) has a Poisson distribution with parameter
\[
\frac{\lambda}{\mu} + \left( \frac{\nu - \lambda}{\mu} \right) e^{-\mu t}.
\]

**Solution:**
(a) Let \( Y(t) \) denote the number of customers arriving in \((0,t]\). Then \( (Y(t))_{t \geq 0} \) is a Poisson process. Thus \( Y(t) \sim \text{Poisson}(\lambda t) \) and given \( Y(t) = k \), the \( k \) arrival times \( T_1, T_2, \ldots, T_k \) are distributed as i.i.d. \( \text{Uniform}(0,t) \). A customer arriving at time \( s \) is still in service at time \( t \geq s \) w.p. \( e^{-\mu(t-s)} \). Thus given \( Y(t) = k \), we have
\[
P(X_1(t) = j | Y(t) = k) = E \sum_{S \subseteq [k], |S| = j} \prod_{i \in S} e^{-\mu(t-T_i)} \prod_{i \notin S} (1 - e^{-\mu(t-T_i)})
\]
\[
= \sum_{S \subseteq [k], |S| = j} \left( E e^{-\mu(t-T_i)} \right)^j \left( 1 - E e^{-\mu(t-T_i)} \right)^{k-j}
\]
\[
= \binom{k}{j} p^j (1 - p)^{k-j}
\]
where \( p = 1/t \int_0^t e^{-\mu s} \, ds = (1 - e^{-\mu t})/\mu t \). Thus
\[
X_1(t) | T(t) = k \sim \text{Bin}(k, (1 - e^{-\mu t})/\mu t).
\]
Hence
\[
X_1(t) \sim \text{Poisson} \left( \frac{\lambda}{\mu} (1 - e^{-\mu t}) \right).
\]
Here we used the fact that, if \( Y \sim \text{Poisson}(a) \) and \( X | Y = n \sim \text{Bin}(n, p) \) then
\[
P(X = k | Y = n) = \sum_{n \geq k} P(Y = n) P(X = k | Y = n) = \sum_{n \geq k} e^{-a} a^n/n! \cdot \binom{n}{k} p^k (1 - p)^{n-k} = e^{-a} e^{ap} k^k / k! \cdot \sum_{n \geq k} (a(1 - p))^{n-k}/(n-k)! = e^{-ap} e^ap = (ap)^k / k! \text{ for } k \geq 0 \text{ and thus } X \sim \text{Poisson}(ap).
\]
(b) We have
\[
p_{ij}(t) = P(\text{Bin}(i, e^{-\mu t}) + \text{Poisson}(\lambda/\mu \cdot (1 - e^{-\mu t})) = j)
\]
\[
= \sum_{k=0}^{\min(i,j)} \frac{i!}{k!} e^{-\lambda/\mu} \frac{(1 - e^{-\mu t})^j - e^{-\lambda/\mu \cdot (1 - e^{-\mu t})}}{(j-k)!} \frac{(1 - e^{-\mu t})^k}{(k)!}
\]
\[
= t e^{-\lambda/\mu \cdot (1 - e^{-\mu t})} \sum_{k=0}^{\min(i,j)} \frac{(1 - e^{-\mu t})^k}{k!} \frac{(1 - e^{-\mu t})^j}{(j-k)!}
\]
(c) Use thinning argument. If \( X \sim \text{Poisson}(\nu) \) and given \( X = k, Y \sim \text{Bin}(k, p) \) then \( Y \sim \text{Poisson}(\nu p) \).
(d) Use the fact that sum of two independent \( \text{Poisson}(\mu) \) and \( \text{Poisson}(\lambda) \) r.v. is a \( \text{Poisson}(\lambda + \mu) \) r.v.

3. (4+(5+6) pts) (a) A stochastic process \((M_n)_{n \geq 0}\) is called a **super-martingale** w.r.t. the filtration \((\mathcal{F}_n)_{n \geq 0}\) if
i) \( E|M_n| < \infty \) for all \( n \geq 0 \).
ii) \( M_n \) is \( \mathcal{F}_n \) measurable for all \( n \geq 0 \) and
iii) \( E(M_{n+1} | \mathcal{F}_n) \leq M_n \) for all \( n \geq 0 \).
Let \( T \) be a stopping time such that \( T < \infty \) a.s. and assume that \( |M_n| \leq C \) for all \( n \leq T \). Prove the Optional Stopping theorem for super-martingales, i.e.,
\[
E(M_T) \leq E(M_0).
\]
4. (3+4+6+2 pts) Let $X = (X_n)_{n \geq 0}$ be a discrete time Markov chain with transition matrix $P = ((p_{ij}))_{ij \in I}$ and finite state space $I$. A function $f$ on $I$ is called a super-harmonic function iff

$$(Pf)(i) = \sum_{j \in I} p_{ij} f(j) \leq f(i) \text{ for all } i \in I.$$ 

Prove that, $X$ is irreducible and recurrent if and only if every super-harmonic function is constant.

**Hint:** For the if part show that $i \mapsto P_i(H(j) < \infty)$ is super-harmonic for all $j$ fixed.

**Solution:** (a) Let $(M_n, \mathcal{F}_n)_{n \geq 0}$ be a super-martingale. Thus we have

$$E(M_{n+1} - M_n \mid \mathcal{F}_n) \leq 0 \text{ for all } n \geq 0.$$ 

Let $T$ be a stopping time such that $T < \infty$ a.s. and assume that $|M_n| \leq C$ for all $n \leq T$. Define the bounded stopping time $T_m = T \wedge m$ for $m \geq 0$. Then we have

$$E(M_{T_m} - M_0) = \sum_{n=0}^{m-1} E((M_{n+1} - M_n) \cdot 1_{\{n < T_m\}}) \leq 0$$

as $\{n < T_m\} \in \mathcal{F}_n$. Thus, $E(M_{T_\wedge m}) \leq E(M_0)$ for all $m \geq 0$. Moreover,

$$|E M_T - E M_{T_\wedge m}| \leq E |M_T - M_{T_\wedge m}| \leq 2C P(T > m) \to 0 \text{ as } m \to \infty,$$

as $T < \infty$ a.s. Thus taking $m \uparrow \infty$ we have

$$E M_T \leq E M_0.$$

(b) Let $X = (X_n)_{n \geq 0}$ be a discrete time Markov chain with transition matrix $P = ((p_{ij}))_{ij \in I}$ and finite state space $I$.

i) Only if part: Let $X$ be irreducible and recurrent. Take a bounded super-harmonic function $f$ on $I$. Define $M_n = f(X_n), \mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$ for $n \geq 0$. It is easy to check that $(M_n, \mathcal{F}_n)_{n \geq 0}$ is a bounded super-martingale as $E(M_{n+1} \mid \mathcal{F}_n) = E(f(X_{n+1}) \mid X_n) = (Pf)(X_n) \leq f(X_n) = M_n$. Using OST for the hitting time $T_j = H(j)$ with $X_0 = i$, we get

$$f(j) = E_i(M_{T_j}) \leq E_i(M_0) = f(i),$$

where we used the fact $T_j < \infty$ a.s. by recurrence. Since $i, j$ are arbitrary, we have $f(j) \leq f(i)$ for all $i, j$. Thus $f(i) = f(j)$ for all $i, j$.

ii) If Part: Fix $j \in I$. For $i \in I$, we have

$$f_{ij} := P_i(T_j < \infty) = \sum_{k \neq j} p_{ik} f_{kj} + p_{ij} \geq \sum_{k \in I} p_{ik} f_{kj}.$$ 

Thus the function $i \mapsto f_{ij}$ is super-harmonic for fixed $j$, and is, thus, a constant, i.e., $f_{ij} = f_{jj} = 1$ for all $i, j$. In particular, $i \mapsto j$ for all $i, j \in I$, proving that $X$ is irreducible. In order to show recurrence, either we can use the fact that any finite irreducible chain is positive recurrent or we can directly show that $P_j(R_j < \infty) = 1$ where $R_j := \inf\{n \geq 1 \mid X_n = j\}$. Note that we have

$$P_j(R_j < \infty) = \sum_{i \in I} p_{ji} P_i(T_j < \infty) = \sum_{i \in I} p_{ji} = 1.$$ 

4. (3+4+6+2 pts) Let $X = (X_t)_{t \geq 0}$ be a birth and death process with state space $I = \mathbb{Z}_+$ and rate matrix $Q = ((q_{ij}))_{i,j \in I}$ where $q_{i,i+1} = \lambda_i, i \geq 0, q_{i,i-1} = \mu_i, i \geq 1$ and $q_{ij} = 0$ for all other $j \neq i$. Rigorously prove the following:

(a) There exists a (unique) invariant distribution if and only if

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty.$$
(b) \( X \) is transient if and only if
\[
\sum_{n=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_n}{\lambda_1 \lambda_2 \cdots \lambda_n} < \infty.
\]

\textbf{Hint:} Use the jump chain to calculate \( h_i = P_i(Y_n = 0 \text{ for some } n \geq 0), i \geq 0. \)

(c) \( X \) is non-explosive if and only if
\[
\sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n \lambda_{n-1}} + \cdots + \frac{\mu_n \mu_{n-1} \cdots \mu_1}{\lambda_n \lambda_{n-1} \cdots \lambda_1 \lambda_0} \right) = \infty.
\]

\textbf{Hint:} Use the result that \( Q \) is non-explosive if and only if \( x = 0 \) is the only bounded solution of \( Qx = x. \)

(d) Given an example of a birth and death chain that is transient and has an unique invariant distribution.

\textbf{Solution:} (a) We recall that a measure \( \pi \) is in detailed balance with \( Q \) iff \( \pi_i q_{ij} = \pi_j q_{ji} \) for all \( i \neq j \in I \). The system of equations read
\[
\begin{align*}
\pi_0 \lambda_0 &= \pi_1 \mu_1 \\
\pi_1 \lambda_1 &= \pi_2 \mu_2 \\
& \vdots \\
\pi_n \lambda_n &= \pi_{n+1} \mu_{n+1} \\
& \vdots
\end{align*}
\]
and thus solving we have
\[
\pi_n = \pi_0 \prod_{i=1}^{n+1} \frac{\lambda_{i-1}}{\mu_i}, \quad n \geq 0
\]
where \( \pi_0 > 0 \) is an arbitrary positive constant. Thus there exists a measure which is in detailed balance with the rate matrix \( Q \). Moreover, the invariant distribution exists iff
\[
\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty.
\]
The invariant distribution is unique as the chain is irreducible.

(b) Let \( Y_n \) be the discrete time jump chain with transition matrix \( P = ((p_{ij}))_{ij \geq 0} \) where
\[
p_{0,1} = 1, \quad p_{i,i+1} = p_i = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad p_{i,i-1} = q_i = \frac{\mu_i}{\lambda_i + \mu_i} \text{ for } i \geq 1, \quad \text{and } 0 \text{ otherwise.}
\]

Define \( h_i = P_i(T_0 < \infty) \) where \( T_0 = \inf\{n \geq 0 \mid Y_n = 0\} \) is the hitting time for 0 and \( i \geq 0. \) Clearly, \((h_i)_{i \geq 0}\) is the minimal non-negative solution to the system of equations
\[
\begin{align*}
h_0 &= 1, \\
h_n &= p_n h_{n+1} + q_n h_{n-1} \text{ for } n \geq 1.
\end{align*}
\]
Defining \( \Delta h_n := h_n - h_{n-1}, n \geq 1 \) we have
\[
\Delta h_{n+1} = \frac{q_n}{p_n} \Delta_n = \frac{\mu_n}{\lambda_n} \Delta h_n = \cdots = \prod_{i=1}^{n} \frac{\mu_i}{\lambda_i} \cdot \Delta h_1.
\]

Summing we have
\[
h_{n+1} = 1 - (1 - h_1) \sum_{i=1}^{n} \prod_{j=1}^{i} \frac{\mu_j}{\lambda_j}, \quad n \geq 0.
\]
If
\[
\sum_{n=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_n}{\lambda_1 \lambda_2 \cdots \lambda_n} = \infty,
\]
the only solution is \( h_0 = h_1 = 1 = h_n \) for all \( n \geq 0 \). Thus \( P_0(R_0 < \infty) = P_1(T_0 < \infty) = 1 \) and the chain is recurrent. On the other hand, if

\[
A := \sum_{n=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_n}{\lambda_1 \lambda_2 \cdots \lambda_n} < \infty,
\]

the minimal solution is given by (as \( h_{n+1} \geq 0 \) for all \( n \) implies that \( h_1 \geq 1 - 1/A \))

\[
h_1 = 1 - \frac{1}{A}, h_{n+1} = 1 - \frac{1}{A} \sum_{i=0}^{n} \frac{\mu_i}{\lambda_i}, \quad n \geq 1.
\]

Thus \( P_0(R_0 < \infty) = P_1(T_0 < \infty) = 1 - 1/A < 1 \) and the chain is transient.

(c) We will use the result that \( Q \) is non-explosive if and only if \( x = 0 \) is the only bounded solution of \( Qx = x \). We solve the system \( x = Qx; x = (x_i)_{i \geq 0} \). Without loss of generality we can assume that \( x_0 \geq 0 \), otherwise we can work with \(-x\). The equation reads

\[
x_0 = -\lambda_0 x_0 + \lambda_0 x_1 \\
x_i = \mu_i x_{i-1} - (\mu_i + \lambda_i) x_i + \lambda_i x_{i+1}, \quad \text{for } i \geq 1.
\]

Define \( \Delta x_i = x_i - x_{i-1}, i \geq 1 \). Thus we have

\[
\Delta x_1 = \frac{x_0}{\lambda_0} \\
\Delta x_{i+1} = \frac{\mu_i}{\lambda_i} \cdot \Delta x_i + \frac{x_i}{\lambda_i}, \quad i \geq 1.
\]

By induction, it is now easy to see that \( x_{i+1} \geq x_i \geq 0 \) for all \( i \geq 0 \). Moreover, the solution always exists as

\[
x_1 = \left(1 + \frac{1}{\lambda_0}\right) x_0 \\
x_2 = \frac{\mu_1}{\lambda_1} \cdot \frac{x_0}{\lambda_0} + \left(1 + \frac{1}{\lambda_1}\right) \left(1 + \frac{1}{\lambda_0}\right) x_0, \\
x_3 = \ldots
\]

and \( x = 0 \) iff \( x_0 = 0 \).

Now, we have

\[
\Delta x_{i+1} = \frac{x_i}{\lambda_i} + \frac{\mu_i}{\lambda_i} \cdot \Delta x_i \\
= \frac{x_i}{\lambda_i} + \frac{\mu_i}{\lambda_i} \cdot \frac{x_{i-1}}{\lambda_{i-1}} + \frac{\mu_i}{\lambda_i} \cdot \frac{\mu_{i-1}}{\lambda_{i-1}} \cdot \Delta x_{i-1} \\
= \ldots \\
= \frac{x_i}{\lambda_i} + \frac{\mu_i}{\lambda_i} \cdot \frac{x_{i-1}}{\lambda_{i-1}} + \frac{\mu_i}{\lambda_i} \cdot \frac{\mu_{i-1}}{\lambda_{i-1}} \cdot \frac{x_{i-2}}{\lambda_{i-2}} + \ldots + \frac{\mu_i}{\lambda_i} \cdot \frac{\mu_{i-1}}{\lambda_{i-1}} \cdots \frac{x_0}{\lambda_0} \\
\geq \left( \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i \lambda_{i-1}} + \frac{\mu_i \mu_{i-1}}{\lambda_i \lambda_{i-1} \lambda_{i-2}} + \ldots + \frac{\mu_i \mu_{i-1} \cdots \mu_1}{\lambda_i \lambda_{i-1} \cdots \lambda_0} \right) x_0.
\]

If

\[
B := \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n \lambda_{n-1}} + \ldots + \frac{\mu_n \mu_{n-1} \cdots \mu_1}{\lambda_n \lambda_{n-1} \cdots \lambda_1 \lambda_0} \right) = \infty,
\]

and \( x_0 > 0 \) then \( x_n = x_0 + \sum_{i=1}^{n} \Delta x_i \to \infty \) as \( n \to \infty \), and the solution \( (x_i)_{i \geq 0} \) is unbounded. Hence, the only bounded solution of \( x = Qx \) is \( x = 0 \), implying that \( Q \) is non-explosive.

Conversely, from the fact that \( x_i 's \) are increasing we have

\[
\Delta x_{i+1} \leq \left( \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i \lambda_{i-1}} + \frac{\mu_i \mu_{i-1}}{\lambda_i \lambda_{i-1} \lambda_{i-2}} + \ldots + \frac{\mu_i \mu_{i-1} \cdots \mu_1}{\lambda_i \lambda_{i-1} \cdots \lambda_0} \right) x_i.
\]
and
\[ x_{i+1} = x_i + \Delta x_{i+1} \leq \left( 1 + \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i \lambda_{i-1}} + \frac{\mu_i \mu_{i-1}}{\lambda_i \lambda_{i-1} \lambda_{i-2}} + \cdots + \frac{\mu_i \mu_{i-1} \cdots \mu_1}{\lambda_i \lambda_{i-1} \lambda_{i-2} \cdots \lambda_0} \right) x_i \]
\[ \leq \exp\left( \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i \lambda_{i-1}} + \frac{\mu_i \mu_{i-1}}{\lambda_i \lambda_{i-1} \lambda_{i-2}} + \cdots + \frac{\mu_i \mu_{i-1} \cdots \mu_1}{\lambda_i \lambda_{i-1} \lambda_{i-2} \cdots \lambda_0} \right) x_i \]
for all \( i \geq 0 \). Here we used the fact that \( 1 + x \leq e^x \) for \( x \geq 0 \). By iterating, we get
\[ x_{n+1} \leq \exp\left[ \sum_{i=0}^{\infty} \left( \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i \lambda_{i-1}} + \frac{\mu_i \mu_{i-1}}{\lambda_i \lambda_{i-1} \lambda_{i-2}} + \cdots + \frac{\mu_i \mu_{i-1} \cdots \mu_1}{\lambda_i \lambda_{i-1} \lambda_{i-2} \cdots \lambda_0} \right) \right] x_0 \]
for all \( n \geq 0 \).

Hence, if the sum in the exponent is finite, then \((x_i)_{i \geq 0}\) is bounded, implying that \( Q \) is explosive.

(d) Take \( \lambda_0 = p, \lambda_i = a^i p, \mu_i = a^i (1 - p), i \geq 1 \) where \( a > 1, p \in (1/2, a/(1 + a)) \) so that \((1 - p)/p < 1\) and \( p/a(1 - p) < 1\). Then we have
\[ \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} = \sum_{n=1}^{\infty} \frac{p^n}{a^n (1 - p)^n} < \infty \]
and
\[ \sum_{n=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_n}{\lambda_1 \lambda_2 \cdots \lambda_n} = \sum_{n=1}^{\infty} \frac{(1 - p)^n}{p^n} < \infty. \]

Thus the chain is transient but has a unique invariant distribution.

5. (10 pts) Let \((X_n)_{n \geq 0}\) be a discrete-time Markov chain on the state space \( I \). Let \( B(i) \) be the collection of bounded real-valued functions on \( I \). For each \( \alpha \in (0, 1) \), define \( U_\alpha : B(I) \rightarrow B(I) \) as
\[ (U_\alpha f)(i) := E_i \left( \sum_{n=0}^{\infty} \alpha^n f(X_n) \right), \quad i \in I. \]

Take \( \alpha, \beta \in (0, 1), \alpha \neq \beta \). Show that there exist constants \( c^{(1)}_{\alpha, \beta}, c^{(2)}_{\alpha, \beta} \) in \( \mathbb{R} \) such that
\[ (U_\alpha (U_\beta f))(i) = c^{(1)}_{\alpha, \beta} \cdot (U_\alpha f)(i) + c^{(2)}_{\alpha, \beta} \cdot (U_\beta f)(i), \quad i \in I. \]

In other words, we have
\[ U_\alpha U_\beta = c^{(1)}_{\alpha, \beta} \cdot U_\alpha + c^{(2)}_{\alpha, \beta} \cdot U_\beta. \]

Find the constants.

**Solution:** We consider a function \( f \) on \( I \) as a column vector. Note that for \( n \geq 0, i \in I \) we have
\[ E_i(f(X_n)) = (P^n f)(i). \]

Thus we have, for \( \alpha \in (0, 1) \)
\[ (U_\alpha f)(i) := E_i \left( \sum_{n=0}^{\infty} \alpha^n f(X_n) \right) = \sum_{n=0}^{\infty} \alpha^n (P^n f)(i) \]
or
\[ U_\alpha f = \sum_{n=0}^{\infty} \alpha^n P^n f. \]
Thus we have

\[ U_\alpha(U_\beta f) = \sum_{n=0}^{\infty} \alpha^n P^n(U_\beta f) = \sum_{n=0}^{\infty} \alpha^n P^n \left( \sum_{m=0}^{\infty} \beta^m f \right) \]

\[ = \sum_{n,m=0}^{\infty} \alpha^n \beta^m P^{n+m} f \]

\[ = \sum_{k=0}^{\infty} \left( \sum_{n=0}^{k} \alpha^n \beta^{k-n} \right) P^k f \]

\[ = \sum_{k=0}^{\infty} \frac{\beta^{k+1} - \alpha^{k+1}}{\beta - \alpha} P^k f \]

\[ = \frac{\beta}{\beta - \alpha} \sum_{k=0}^{\infty} \beta^k P^k f + \frac{\alpha}{\alpha - \beta} \sum_{k=0}^{\infty} \alpha^k P^k f = \frac{\beta U_\beta f + \alpha U_\alpha f}{\beta - \alpha}. \]

Here we used the geometric series summation formula \( \sum_{n=0}^{\infty} r^n = (1 - r^{k+1})/(1 - r). \)