1. (2+2 pts) Let $Q$ be the rate matrix of a finite state space Markov chain and $(P(t))_{t \geq 0}$ be the corresponding semigroup of transition matrices.

(a) Show that $Qx = \lambda x$ for $\lambda > 0$ implies that $x = 0$, so $Q$ does not have strictly positive eigenvalues. Conclude that $\lambda I - Q$ is invertible for every $\lambda > 0$.

(b) For $\lambda > 0$, define $R(\lambda) = (r_{ij}(\lambda))_{i,j}$ where

$$r_{ij}(\lambda) = \int_0^\infty e^{-\lambda t}p_{ij}(t)dt.$$

Prove that $R(\lambda) = (\lambda I - Q)^{-1}$. This rigorously proves that $\int_0^\infty e^{-\lambda t}e^{Q}dt = (\lambda I - Q)^{-1}$.

**Solution**: (a) Assume that, $Qx = \lambda x$ for some $\lambda > 0, x \neq 0$. Let $i^* = \arg\max \{|x_i| \mid i \in I\}$. It exists since the state space $I$ is finite. W.l.o.g. we can assume that $x_{i^*} > 0$ (by working with $-x$ instead of $x$ if needed). Now, we have

$$\lambda x_{i^*} = \sum_{i \in I} q_{i^*,i}x_i = -q_{i^*,i^*}x_{i^*} + \sum_{i \neq i^*} q_{i^*,i}x_i = -\sum_{i \neq i^*} q_{i^*,i}(x_{i^*} - x_i).$$

Note that the LHS $> 0$, whereas the RHS is $\leq 0$. Contradiction!

Thus for $\lambda > 0$, $(\lambda I - Q)x = 0$ implies that $x = 0$. Hence $(\lambda I - Q)$ is invertible.

(b) It is enough to prove that $R(\lambda)(\lambda I - Q) = I$. We have for $i, j \in I$

$$(R(\lambda)(\lambda I - Q))_{ij} = \sum_{k \in I} r_{ik}(\lambda)(\lambda \delta_{kj} - q_{kj})$$

$$= \lambda r_{ij}(\lambda) - \sum_{k \in I} \int_0^\infty e^{-\lambda t}p_{ik}(t)q_{kj}dt$$

$$= \lambda r_{ij}(\lambda) - \int_0^\infty e^{-\lambda t} \sum_{k \in I} p_{ik}(t)q_{kj}dt \quad \text{(by Fubini)}$$

$$= \lambda r_{ij}(\lambda) - \int_0^\infty e^{-\lambda t}p'_{ij}(t)dt \quad \text{(by Forward equations)}$$

$$= \lambda r_{ij}(\lambda) - e^{-\lambda t}p_{ij}(t)|_0^\infty - \int_0^\infty \lambda e^{-\lambda t}p_{ij}(t)dt$$

$$= \lambda r_{ij}(\lambda) - (0 - p_{ij}(0)) - \lambda r_{ij}(\lambda)$$

$$= \delta_{ij}.$$

This completes the proof.

2. (3 pts) Two fleas are bound together to take part in a race on the vertices $A, B, C$ of a triangle. Flea 1 hops at random times in the clockwise direction; each hop takes the pair from one vertex to the next and the times between successive hops of Flea 1 are independent random variables, each with with exponential distribution, rate 1. Flea 2 behaves similarly, but hops in the anti-clockwise direction, the times between his hops having same rate 1. Show that the probability that they are at $A$ at a given time $t > 0$ (starting from $A$ at time $t = 0$) is

$$\frac{1}{3}(1 + 2e^{-3t}).$$
3. (3 pts) Let $(\lambda, n \geq 0)$ be a sequence of Probability Distributions on a countable state space $I$ such that

$$\lambda_n(i) \to \mu(i) \text{ as } n \to \infty$$

for all $i \in I$. Assume that $\mu$ is also a Probability Distribution, so that $\sum_{i \in I} \mu(i) = 1$. Prove that, the total-variation distance between $\lambda_n$ and $\mu$ converges to 0 as $n \to \infty$.

**Solution:** Given $\varepsilon > 0$, we can choose a finite set $F_\varepsilon \subset I$ such that

$$\sum_{i \notin F_\varepsilon} \mu(i) \leq \varepsilon/4.$$  

Since, $F_\varepsilon$ is finite and $\lambda_n(i) \to \mu(i)$ as $n \to \infty$ for all $i \in F_\varepsilon$, we can find $N$ large such that

$$\max_{i \in F_\varepsilon} |\lambda_n(i) - \mu(i)| \leq \varepsilon/4 \text{ for all } n \geq N.$$  

Now we have, for all $n \geq N$

$$||\lambda_n - \mu||_{TV} = \sum_{i \in I} |\lambda_n(i) - \mu(i)| \leq \sum_{i \notin F_\varepsilon} |\lambda_n(i) - \mu(i)| + \sum_{i \in F_\varepsilon} (\lambda_n(i) + \mu(i))$$

$$\leq \sum_{i \notin F_\varepsilon} |\lambda_n(i) - \mu(i)| + \sum_{i \notin F_\varepsilon} (\lambda_n(i) - \mu(i)) + 2 \sum_{i \notin F_\varepsilon} \mu(i)$$

$$= \sum_{i \notin F_\varepsilon} |\lambda_n(i) - \mu(i)| + \sum_{i \notin F_\varepsilon} (\mu(i) - \lambda_n(i)) + 2 \sum_{i \notin F_\varepsilon} \mu(i)$$

$$\leq 2 \sum_{i \notin F_\varepsilon} |\lambda_n(i) - \mu(i)| + 2 \sum_{i \notin F_\varepsilon} \mu(i)$$

$$\leq \varepsilon.$$  

Since $\varepsilon$ is arbitrary, we have $||\lambda_n - \mu||_{TV} \to 0$ as $n \to \infty$.

4. (2+3 pts) Consider the biased random walk $X_n, n \geq 0$ on $\mathbb{Z}$ with transition matrix $p_{i,i+1} = \lambda, p_{i,i-1} = \mu, \lambda + \mu = 1, \lambda \neq \mu$ and $p_{ij} = 0$ if $|i-j| > 1$ and $X_0 = 0$.

(a) Show that $M_n = (\mu/\lambda)^{X_n}, n \geq 0$ is a martingale w.r.t. the usual filtration.

(b) Fix $a, b \in \{1, 2, \ldots\}$. Find

$$\mathbb{P}_0(X_n \text{ hits } -a \text{ before hitting } b).$$

**Solution:** (a) $M_n$ is a bounded r.v. with $M_n \in [0, \max\{(\mu/\lambda)^{-n}, (\mu/\lambda)^n\}]$ and hence is integrable. Also, $M_n$ is adapted to $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$. Finally, it is easy to check that (check!) $\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = M_n$ for all $n \geq 0$. Hence $M_n, n \geq 0$ is a martingale w.r.t. the usual filtration $\{\mathcal{F}_n, n \geq 0\}$.

(b) Using Optional stopping theorem condition (ii) with the stopping time $T = \inf\{n \geq 0 \mid X_n \in \{-a, b\}\}$ (Check the condition!) we get

$$1 = \mathbb{E}(M_0) = \mathbb{E}(M_T) = (\mu/\lambda)^{-a} \mathbb{P}(X_T = -a) + (\mu/\lambda)^b \mathbb{P}(X_T = b).$$

Solving we get

$$\mathbb{P}_0(X_n \text{ hits } -a \text{ before hitting } b) = \mathbb{P}(X_T = -a) = 1 - \mathbb{P}(X_T = b) = \frac{1 - (\mu/\lambda)^b}{(\mu/\lambda)^{-a} - (\mu/\lambda)^b}.$$
5. (3 pts) Consider a fleet of \( N \) buses. Each bus breaks down independently at rate \( \mu \), when it is sent to the depot for repair. The repair shop can only repair one bus at a time and each bus takes an exponential time of parameter \( \lambda \) to repair. Find the equilibrium distribution of the number of buses in service.

**Solution:** (Brief solution) The chain is an irreducible and positive recurrent birth and death chain. Using Detailed Balance for the rate matrix

\[
Q = \begin{bmatrix}
-\lambda & \lambda & 0 & 0 & \ldots & 0 & 0 \\
\mu & -(\lambda + \mu) & \lambda & 0 & \ldots & 0 & 0 \\
0 & 2\mu & -(\lambda + 2\mu) & \lambda & \ldots & 0 & 0 \\
0 & 0 & 3\mu & -(\lambda + 3\mu) & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ldots & -(\lambda + (N-1)\mu) & \lambda \\
0 & 0 & 0 & 0 & \ldots & -N\mu & -N\mu
\end{bmatrix}
\]

where \( q_{i,i+1} = \lambda \) for \( i = 0, 1, \ldots, N - 1; q_{i,i-1} = i\mu \) for \( i = 1, 2, \ldots, N \) and \( q_{ii} = -(\lambda + i\mu) \) \( i = 0, 1, 2, \ldots, N \). We get the measure

\[
\pi_i = \lambda_0 \cdot \frac{\lambda^i}{i!}, i = 0, 1, \ldots, N
\]

which satisfies \( \lambda_i q_{i,i-1} = \lambda_{i-1} q_{i-1,i} \) for all \( i = 1, 2, \ldots, N \). Normalizing we get the equilibrium distribution as

\[
\pi_i = \frac{1}{\mathbb{E}} \left( \frac{\lambda}{\mu} \right)^i \left( \sum_{k=0}^{N} \frac{1}{k!} \left( \frac{\lambda}{\mu} \right)^k \right)^{-1}, i = 0, 1, \ldots, N.
\]

6. (4 pts) Harry’s restaurant is on a beach and the number of customers highly depends on weather conditions. According to Harry, days are either cloudy (C), sunny (S) or rainy (R). After longtime observations, Harry came to the conclusion that weather can be modeled by a continuous-time Markov chain \((X_t)_{t \geq 0}\) with generator matrix (the order of states is C, S, R, and the rates are in days)

\[
Q = \begin{bmatrix}
-\frac{3}{2} & 0 & 0 \\
\frac{3}{2} & -\frac{1}{3} & \frac{1}{6} \\
\frac{1}{3} & \frac{1}{6} & -\frac{1}{3}
\end{bmatrix}
\]

(a) Calculate the expected duration of cloudy weather.

(b) Find the stationary distribution of \( X \).

(c) Weather is cloudy. Calculate the expected number of days until the next sunny period.

(d) We have just entered a sunny period. Calculate the expected number of days until the next sunny period.

**Solution:**

(a) Duration of cloudy weather is exponentially distributed with rate 1/5 with mean 5.

(b) Solving for \( \pi Q = 0 \) we get \( \pi = \frac{1}{221}(105, 80, 36) \).

(c) \( E_C(T_S) = 5 + \frac{1}{2} E_R(T_S) \), \( E_R(T_S) = 3 + \frac{1}{2} E_C(T_S) \). Solving we get \( E_C(T_S) = 36/5 = 7.2 \).

(d) \( m_S = 1/(q_S \pi S) = 221/20 = 11.05 \).

7. (1+1+2+3 pts) Let \((X_t)_{t \geq 0}\) be a Markov chain on the integers \( \mathbb{Z} \) with transition rates \( q_{i,i+1} = \lambda q_i \), \( q_{i,i-1} = \mu q_i \), and \( q_{ij} = 0 \) if \( |j - i| > 1 \), where \( \lambda + \mu = 1 \) and \( q_i > 0 \) for all \( i \).

(a) In the case where \( \mu = 0 \), write down a necessary and sufficient condition for \((X_t)_{t \geq 0}\) to be explosive.

(b) Why is this condition necessary for \((X_t)_{t \geq 0}\) to be explosive for all \( \mu \in [0, 1/2]\)?

(c) Show that, in general, \((X_t)_{t \geq 0}\) is non-explosive if

one of the following conditions holds: (i) \( \lambda = \mu; \)

(ii) \( \lambda > \mu \) and \( \sum_{i=1}^{\infty} 1/q_i = \infty; \)

(iii) \( \lambda < \mu \) and \( \sum_{i=1}^{\infty} 1/q_i = \infty. \)
(d) Show that, the condition in part (c) is also necessary for non-explosiveness.

**Hint:** When \( \lambda \neq \mu \), for the jump chain \( \sup_{i \in \mathbb{Z}} \mathbb{E}_0 [\{ n \geq 0 \mid Y_n = i \}] \leq C \) for some finite constant \( C \), that is the jump chain spends finite time (uniformly bounded in expectation) in any state.

**Solution:** W.l.o.g. we can take \( X_0 = 0 \).

(a) Assume \( \mu = 0 \). In this case, the CTMC is a pure birth process. The jump chain moves to the right and the waiting time \( W_i \) at \( i \geq 0 \) is exponential of rate \( q_i \). Then \( \zeta = \sum_{i=0}^{\infty} W_i \) is a sum of independent exponential random variables and \( \zeta < \infty \) iff \( \sum_{i=1}^{\infty} 1/q_i < \infty \).

(b) If \( \mu \in [0,1/2) \), then \( \lambda > \mu \) and for the jump chain \( Y_n \to +\infty \) a.s. Therefore, \( X_i \to \infty \) a.s., and it particular, \( X_i \) passes through every \( i \geq 1 \). Hence \( \zeta \geq \sum_{i=0}^{\infty} W_{\sigma_i} \), where \( W_{\sigma_i}, i \geq 1 \) is a sequence of independent exponential random variables, with \( W_{\sigma_i} \sim \text{Exponential with rate } q_i \). If \( \sum_{i=1}^{\infty} 1/q_i = \infty \), then also \( \sum_{i=0}^{\infty} W_{\sigma_i} = \infty \) and therefore \( \zeta = \infty \). Thus, \( \zeta < \infty \) implies that \( \sum_{i=1}^{\infty} 1/q_i < \infty \).

(c) (i) If \( \lambda = \mu \), the jump chain is recurrent and hence \( (X_t)_{t \geq 0} \) is non-explosive. (ii) \( \lambda > \mu \) and \( \sum_{i=1}^{\infty} 1/q_i = \infty \) is equivalent to part (b). (iii) \( \lambda < \mu \) and \( \sum_{i=1}^{\infty} 1/q_{-i} = \infty \) is same as (ii) but in the negative axis direction.

(d) We assume that the chain is non-explosive, i.e., \( \zeta = \infty \). If \( \lambda \neq \mu \), we have nothing to prove. So assume either \( \lambda > \mu \) or \( \lambda < \mu \). Since these two cases are symmetric, we give the proof for \( \lambda > \mu \). It remains to show that \( \sum_{i=1}^{\infty} 1/q_i = \infty \).

Suppose, on the contrary, that

\[
\sum_{i=1}^{\infty} 1/q_i < \infty.
\]

The jump chain \( Y_n \to \infty \) a.s. and hence \( X_i \to \infty \) a.s. Thus \( T_+ = \int_0^\zeta \mathbf{1}_{\{X_i \leq 0\}} \, dt \), the total time \( X \) spends in the negative states, is finite a.s.

Let \( T_+ = \int_0^\zeta \mathbf{1}_{\{X_i > 0\}} \, dt \) be the total time \( X \) spends in the positive states. Note that \( T = \sum_{i=1}^{\infty} T_i \) where \( T_i = \int_0^{\zeta} \mathbf{1}_{\{X_i = i\}} \, dt \) for \( i \geq 1 \) and \( \mathbb{E}_0(T_i) = 1/q_i \mathbb{E}_0(\{ n \geq 0 \mid Y_n = i \}) \) (Since each time the jump chain visits the state \( i \) the \( X \) chain spends Exponential rate \( q_i \) time there). Hence if we show that

\[
\sup_{i \geq 1} \mathbb{E}_0(\{ n \geq 0 \mid Y_n = i \}) \leq C,
\]

then

\[
\mathbb{E}(T_+) = \sum_{i \geq 1} \mathbb{E}(T_i) = \sum_{i \geq 1} 1/q_i \mathbb{E}_0(\{ n \geq 0 \mid Y_n = i \}) \leq C \sum_{i \geq 1} 1/q_i < \infty.
\]

This is a contradiction, since \( T_+ + T_- = \zeta = \infty \).

Now, to complete the proof we note that for \( i \geq 1 \),

\[
\mathbb{E}_0(\{ n \geq 0 \mid Y_n = i \}) = \mathbb{E}_0(\{ n \geq 0 \mid Y_n = i \}) \quad \text{(by irreducibility)}
\]

\[
= \mathbb{E}_0(\{ n \geq 0 \mid Y_n = 0 \}) \quad \text{(by symmetry)}
\]

which is finite, since the jump chain \( Y_n \) is transient. This completes the proof.