Homework 5
MATH 564/STAT 555: Applied Stochastic Processes
Due date: November 9, 2017

1. (3+1 pts.) (a) Which of the following matrices is the exponential of a $Q$-matrix?

   $(a) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  
   $(b) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  
   $(c) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (b) What consequences do your answers have for the discrete-time Markov chains with these transition matrices?

2. (2+3 pts.) Arrivals of the 1N bus form a Poisson process of rate three buses per hour, and arrivals of the 2U bus form an independent Poisson process of rate five buses per hour.

   (a) What is the probability that exactly four buses pass by in one hour?
   (b) What is the probability that exactly three 2U buses pass by while I am waiting for a 1N bus?

3. (2 pts.) Given a finite $m \times m$ matrix $Q$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ and a corresponding right-eigenbasis $v_1, v_2, \ldots, v_m$, write $e^{tQ}$ in terms of $\lambda_i, v_i$'s. Explain.

4. (3+1+2 pts.) Assume that $T_1, T_2, \ldots, T_n$ are independent RVs and $T_k \sim$ Exponential with rate $\lambda_k$. Let $T := \min_{k=1,2,\ldots,n} T_k$.

   (a) Show that $T$ is exponentially distributed and compute its rate.
   (b) Find $\mathbb{P}(T_1 < T_2)$.
   (c) Using (a)+(b), calculate $\mathbb{P}(T = T_k)$.

5. (4+1 pts.) Harry’s restaurant is well known for serving great food, but is filthy and hence not for those with weak stomach. During rush hour, customers arrive at the restaurant according to a Poisson process $(X_t)_{t \geq 0}$ of rate $\lambda$, here $X_t$ is the number of customers that arrived on or before time $t$.

   (a) Customers peek in the door and, independently of each other, with probability $q \in (0,1)$ they decide the filth is not for them and depart; with probability $p = 1 - q$ they enter and eat. Let $(Y_t)_{t \geq 0}$ be the process describing customers that are brave enough to enter and eat. Prove that $(Y_t)_{t \geq 0}$ is a Poisson process and determine its rate.

   **Hint:** Let $T_1, T_2, \ldots, T_n$ be i.i.d. Exponential with rate $\lambda$. The density of $J_n := T_1 + T_2 + \cdots + T_n$ is

   \[
   f_n(x) := \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \cdot 1_{x>0}.
   \]

   (b) Assume now that every second/even numbered customer enter and eats, and let $(Z_t)_{t \geq 0}$ be the corresponding process. Is $(Z_t)_{t \geq 0}$ a Poisson process? Explain.

6. (2+1 pts.) Recall the explosion time of a CTMC is $\zeta := \lim_{n \to \infty} J_n$, where $J_n$’s are the successive jump times.

   (a) Give an example of a CTMC on a state space $I$ and a state $i \in I$ such that $\mathbb{P}_i(\zeta < \infty) = 1/2$.
   (b) Generalize the example in part (a) from 1/2 to an arbitrary number $p \in (0,1)$.