1. (5 pts.) In each of the following cases determine whether the stochastic matrix $P$ on the state space $I$, which you may assume irreducible, is reversible:
   
   (a) Let $P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$
   
   (b) Let $P = \begin{bmatrix} 0 & 1-p \\ 1-p & 0 \end{bmatrix}$
   
   (c) $I = \{0, 1, \ldots, N\}$ and $p_{ij} = 0$ if $|j - i| \geq 2$ and $p_{ij} > 0$ if $|j - i| \leq 1$.
   
   (d) $I = \{0, 1, \ldots\}$ and $p_{01} = 1, p_{i,i+1} = p, p_{i,i-1} = 1 - p$ for $i \geq 1$.

   Solution: The stochastic matrix $P$ is reversible if and only if there exists a distribution $\lambda$ such that $P$ and $\lambda$ are in detailed balance: $\lambda_i p_{ij} = \lambda_j p_{ji}$ for all $i, j$.

   (a) $P$ is reversible w.r.t. $\lambda = (b/(a + b), a/(a + b))$.

   (b) The detailed balance equations $\lambda_1 p = \lambda_2 (1 - p), \lambda_2 p = \lambda_3 (1 - p), \lambda_3 p = \lambda_1 (1 - p)$ have a solution if and only if $p = \frac{1}{2}$. Hence, $P$ is reversible only when $p = \frac{1}{2}$.

   (c) The detailed balance equations are $\lambda_i p_{i,i+1} = \lambda_{i+1} p_{i+1,i}$ and have a unique solution

   $$\lambda_k = \lambda_0 \prod_{i=0}^{k-1} \frac{p_{i,i+1}}{p_{i+1,i}}, \quad k = 1, 2, \ldots, N.$$ 

   This can be normalized to a probability distribution, hence $P$ is reversible.

   (d) A unique solution to the detailed balance equations is $\lambda_i = \frac{1}{q} \left( \frac{q}{2} \right)^i \lambda_0, i \geq 0$ where $q = 1 - p$. In case $p < q$ or $p < \frac{1}{2}$, this measure can be normalized, hence $P$ is reversible. When $p \geq \frac{1}{2}$, $P$ is not reversible.

   (e) The measure $\lambda_i = 1, \text{all } i \in I$, is a unique solution of the detailed balance equations. If the state space is finite, it can be normalized, and then $P$ is reversible. If the state space is infinite, there is no finite measure in detailed balance with $P$, so $P$ is not reversible.

2. (2+3 pts.) Let $(X_n)_{n \geq 0}$ be an irreducible Markov chain on the state space $I$ having transition matrix $P$ and an invariant distribution $\pi$. For $A \subseteq I$ let $(Y_n)_{n \geq 0}$ be the random process on $A$ obtained by observing $(X_n)_{n \geq 0}$ whilst in $A$. More precisely, if $T_0 = \min\{n \geq 0 : X_n \in A\}$, and for $m \geq 1, T_m = \min\{n > T_{m-1} : X_n \in A\}$, let $Y_m = X_{T_m}$.

   (a) Prove that $(Y_m)_{m \geq 0}$ is a Markov chain and compute its transition probabilities in terms of $P$.

   (b) Show that $(Y_m)_{m \geq 0}$ is positive recurrent and find its invariant distribution.

   Solution: (a) Since $(X_n)_{n \geq 0}$ is irreducible and has an invariant distribution, it is positive recurrent. In particular, $P(T_m < \infty) = 1$. By the strong Markov property at $T_m$ we have

   $$P(Y_{m+1} = i_{m+1} \mid Y_0 = i_0, Y_1 = i_1, \ldots, Y_m = i_m) = P(X_{T_{m+1}} = i_{m+1} \mid X_{T_0} = i_0, X_{T_1} = i_1, \ldots, X_{T_m} = i_m) = P_{i_m}(X_{T_1} = i_{m+1}).$$

   Define

   $$q_{i,j} := \begin{cases} P_i(X_{T_1} = j) & \text{if } i \in A, j \in A \\ P_i(X_{T_0} = j) & \text{if } i \notin A, j \in A. \end{cases}$$
3. (2+2 pts.) Each morning a student takes one of the $k$ books he owns from his shelf. The probability that he chooses book $i$ is $\alpha_i$, where $0 < \alpha_i < 1$ for $i = 1, \ldots, k$, and choices on successive days are independent. In the evening he replaces the book at the left-hand end of the shelf. Let $p_n$ denote the probability that on day $n$ the student finds the books in the correct order $1, 2, \ldots$ from left to right.

(a) Show that, irrespective of the initial arrangement of the books, $p_n$ converges as $n \to \infty$.

(b) Determine the limit when $k = 3$.

**Solution:** (a) The order of the $k$ books can be represented by a permutation $\sigma$ of $\{1, 2, \ldots, k\}$ where $\sigma_i$ denotes the book number for the $i$-th position from the left, for $i = 1, 2, \ldots, k$ (e.g., $\sigma = (1342)$ for $k = 4$ represents (book 1, book 3, book 4, book 2) in that order from left to right).

The order of books on the shelf at the end of the day can be modeled with a Markov chain $(X_n)_{n \geq 0}$ with the state space $I = S_k$ the set of all permutations of $\{1, 2, \ldots, k\}$ which has transition matrix $P = ((p_{\sigma, \tau}))_{\sigma, \tau \in I}$ where

$$p_{\sigma, \tau} = \begin{cases} \alpha_{\tau_1} & \text{if } \tau = (\sigma_j, \sigma_1, \sigma_2, \ldots, j_{j-1}, \sigma_{j+1}, \ldots, \sigma_k) \text{ for some } j = 1, 2, \ldots, k; \\ 0 & \text{otherwise}. \end{cases}$$

Note that $p_n = P(X_n = \text{id})$ where $\text{id} = (1, 2, 3, \ldots, k)$ is the identity permutation. The chain is clearly irreducible and aperiodic. Since the state space is finite, it is recurrent. Hence it has a unique stationary distribution $\pi = (\pi_{\sigma, \sigma} \in I)$. By the convergence to equilibrium theorem,

$$\lim_{n \to \infty} p_n = \lim_{n \to \infty} P(X_n = \text{id}) = \pi_{\text{id}}.$$
b) We have $|I| = k!$. Thus, when $k = 3$ we have 6 states. One can directly check that the answer is $\frac{\alpha_1 \alpha_2}{1 - \alpha_1}$. However, it is possible to find the invariant distribution for general $k \geq 2$ by using induction to get

$$
\pi_\sigma = \frac{\alpha_{\sigma_1}}{1 - \alpha_{\sigma_1}} \cdot \frac{\alpha_{\sigma_2}}{1 - \alpha_{\sigma_1} - \alpha_{\sigma_2}} \cdots \frac{\alpha_{\sigma_k-1}}{1 - \alpha_{\sigma_1} - \alpha_{\sigma_2} - \cdots - \alpha_{\sigma_{k-1}}} \cdot \alpha_{\sigma_k} \quad \text{for} \quad \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k).
$$

4. (2+1 pts.) Happy Harry used to play semipro basketball where he was a defensive specialist. His scoring productivity per game fluctuated between three states: 1 (scored 0 or 1 points), 2 (scored between 2 and 5 points), 3 (scored more than 5 points). Inevitably, if Harry scored a lot of points in one game, his jealous teammates refused to pass the ball in the next game, so his productivity in the next game was nil. The team statistician, Mrs. Doc, upon observing the transitions between states, concluded these transitions could be modeled by a Markov chain with transition matrix

$$
P = \begin{bmatrix}
0 & \frac{1}{3} & \frac{2}{3} \\
\frac{1}{3} & 0 & \frac{2}{3} \\
1 & 0 & 0
\end{bmatrix}
$$

(a) What is the long run proportion of games that Harry had high scoring game?

(b) The salary structure in the semipro leagues includes incentives for scoring. Harry was paid $500/game for a high scoring performance, $300/game when he scored between 2 and 5 points, and only $100/game when he score nil. What was Harry’s long run earning rate?

**Solution:** (a) We first compute the stationary distribution $\pi = (\pi_1, \pi_2, \pi_3)$ which solves the system $\pi = \pi P$. The solution is $\pi = (9/20, 3/20, 8/20)$. The long run proposition of high scoring games is according to the ergodic theorem equal to

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{X_k = 3\}} = \pi_3 = \frac{8}{20} = \frac{2}{5} \ \text{a.s.}
$$

(b) Again, by Ergodic theorem

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) = \pi_1 f(1) + \pi_2 f(2) + \pi_3 f(3) = \frac{9 \cdot 100 + 3 \cdot 300 + 8 \cdot 500}{20} = 290 \ \text{a.s.}
$$

5. (3 pts.) Given two distributions $\mu, \nu$ on the sample space $I = \{x_1, x_2, \ldots\}$, show that there exists a coupling $(X, Y)$ such that $X \sim \mu, Y \sim \nu$ and

$$
P(X \neq Y) = \frac{1}{2} \sum_{i \geq 1} |\mu(x_i) - \nu(x_i)|.
$$

**Hint:** Enough to find $a(x, y) = P(X = x, Y = y) \in [0, 1]$ for $x, y \in I$ such that $\sum_y a(x, y) = \mu(x), \sum_x a(x, y) = \nu(y)$. Then $P(X \neq Y) = 1 - P(X = Y) = 1 - \sum_x a(x, x)$. Also note that $|x - y| = x + y - 2 \min\{x, y\}$.

**Solution:** Note that $|x - y| = x + y - 2 \min\{x, y\}$. Thus we have

$$
\frac{1}{2} \sum_{i \geq 1} |\mu(x_i) - \nu(x_i)| = \frac{1}{2} \sum_{i \geq 1} (\mu(x_i) + \nu(x_i) - 2 \min\{\mu(x_i), \nu(x_i)\}) = 1 - \sum_{i \geq 1} \min\{\mu(x_i), \nu(x_i)\} = 1 - \lambda
$$

where

$$
\lambda := \sum_{i \geq 1} \min\{\mu(x_i), \nu(x_i)\}.
$$

Let $A_+ := \{x \in I \mid \mu(x) > \nu(x)\}, A_- := \{x \in I \mid \mu(x) < \nu(x)\}$. Now define the following distribution on $I \times I$,

$$
a(x, y) = \begin{cases}
\min\{\mu(x), \nu(x)\}, & \text{if } x = y \\
(\mu(x) - \nu(x))(\nu(y) - \mu(y))/(1 - \lambda), & \text{if } x \in A_+, y \in A_- \\
0, & \text{otherwise.}
\end{cases}
$$

Check that $a$ is a valid distribution with marginals given by $\mu, \nu$. Moreover, if $(X, Y) \sim a$, then $X \sim \mu, Y \sim \nu$, $P(X \neq Y) = 1 - \lambda$. 

6. (2+1 pts.) For a finite irreducible MC with transition matrix $P$ and invariant distribution $\pi$, prove that
\[
\sup_{\lambda} ||\lambda P^n - \pi||_{\text{TV}} = \sup_{x} ||\delta_x P^n - \pi||_{\text{TV}}
\]
and
\[
\sup_{x} ||\delta_x P^n - \pi||_{\text{TV}} \leq \sup_{x,y} ||\delta_x P^n - \delta_y P^n||_{\text{TV}}.
\]

**Solution:** Obviously,
\[
\sup_{\lambda} ||\lambda P^n - \pi||_{\text{TV}} \geq \sup_{x} ||\delta_x P^n - \pi||_{\text{TV}}.
\]

For the other direction, note that for any $\lambda$, we have
\[
||\lambda P^n - \pi||_{\text{TV}} = \sum_{y \in I} \sup_{x \in I} \sum_{x} \lambda_x |p_{x,y}^{(n)} - \pi_y|
\]
\[
\leq \sum_{y \in I} \sup_{x \in I} \sum_{x} \lambda_x \cdot |p_{x,y}^{(n)} - \pi_y|
\]
\[
= \sum_{x \in I} \lambda_x \cdot ||\delta_x P^n - \pi||_{\text{TV}} \leq \sum_{x \in I} \lambda_x \cdot \sup_{y} ||\delta_y P^n - \pi||_{\text{TV}} = \sup_{y} ||\delta_y P^n - \pi||_{\text{TV}}.
\]

Similarly, we have for any $x \in I$,
\[
||\delta_x P^n - \pi||_{\text{TV}} = ||\delta_x P^n - \pi P^n||_{\text{TV}} = \sum_{y \in I} \sup_{z \in I} \sum_{z} \pi_z |p_{z,y}^{(n)} - p_{z,y}^{(n)}|
\]
\[
\leq \sum_{y \in I} \sup_{z \in I} \sum_{z} \pi_z |p_{z,y}^{(n)} - p_{z,y}^{(n)}|
\]
\[
= \sum_{z \in I} \pi_z ||\delta_z P^n - \delta_z P^n||_{\text{TV}} \leq \sup_{z} ||\delta_z P^n - \delta_z P^n||_{\text{TV}}.
\]

7. (3+2+1 pts.) Consider the two state Markov chain with state $\{1, 2\}$ and with transition probabilities $p_{12} = p \in [0, 1]$, $p_{21} = q \in (0, 1)$. Let us assume that $0 < q < 1$ is fixed but we can choose $p \in [0, 1]$ however we like. Moreover, assume that there is a payoff of $r > 0$ every time we visit state 2, and a cost of $c(p) > 0$ every time we visit state 1. Then:

(a) Compute the long-term profit per time step as a function of $p$. Describe a method to find the optimal choice of $p$ to maximize profits. Does such an optimal choice always exist?

(b) Assume that the cost function is linear, i.e., $c(p) = \alpha p$ for some $\alpha > 0$. Show that the optimal choice of $p$ will lead to a profit-per-step of
\[
\left( \frac{r - \alpha q}{1 + q} \right)^+
\]
where we denote $x^+ := \max\{x, 0\}$.

(c) Assume that the cost function is constant, i.e., $c(p) = c$ for all $p$. What is the optimal profit?

**Solution:** Recall that $0 < q < 1$ is fixed. For $p > 0$, the transition matrix $P = \begin{bmatrix} 1 - p & p \\ q & 1 - q \end{bmatrix}$ is irreducible, aperiodic, reversible with stationary distribution $\pi = \frac{q/(p + q), p/(p + q)}$. By convergence to equilibrium theorem, we can write the long term profit per step $\text{profit}(p)$, as a function of $p$ as
\[
\text{profit}(p) = \frac{-qc(p) + pr}{p + q} = r - \frac{q(r + c(p))}{p + q},
\]
here we think cost as negative profit. For $p = 0$, 1 is an absorbing state and 2 $\rightarrow$ 1, thus the equilibrium distribution is $\pi = (1, 0)$ with $\text{profit}(0) = -c(0)$.

To optimize profit, we want to minimize the function
\[
\frac{r + c(p)}{p + q} \text{ for } p \in [0, 1].
\]
We have \( \frac{r + c(p)}{p + q} \geq \frac{r}{1 + q} \), thus we can always find the maximum profit. But, whether the optimal choice of \( p \) exists depends on the continuity of the function \( c(p), p \in [0, 1] \).

(b) When \( c(p) = \alpha p \) for some \( \alpha > 0 \). The function \( \frac{r + \alpha p}{p + q} = \alpha + \frac{r - \alpha q}{p + q} \) is minimized at \( p = 1 \) if \( r \geq \alpha q \) and at \( p = 0 \) otherwise. Thus the maximum profit is

\[
\left( \frac{r - \alpha q}{1 + q} \right)^+ 
\]

where we denote \( x^+ := \max\{x, 0\} \).

(c) If \( c(p) = c > 0 \) for all \( p \), the function \( \frac{r + c}{p + q} \) is minimized at \( p = 1 \) with maximum profit

\[
\frac{r - cq}{1 + q}.
\]