Homework 2

MATH 564/STAT 555: Applied Stochastic Processes

Due date: September 28, 2017

1. (3 pts.) Suppose that \((X_n)_{n \geq 0}^\infty\) is Markov\((\lambda, P)\). Define \(Y_n = X_{kn}\) for some \(k \geq 1\). Show that \((Y_n)_{n \geq 0}^\infty\) is also a Markov chain. Find the initial distribution and transition matrix.

**Solution:** The assumption that \((X_n)_{n \geq 0}^\infty\) is Markov\((\lambda, P)\) implies that
\[
P(X_i = x_i, i = 0, 1, \ldots, n) = \lambda_{x_0} \prod_{i=1}^{n} p_{x_{i-1} x_i},
\]
for all \(n \geq 1, x_0, x_1, \ldots, x_n \in I\) where \(I\) is the state space and \(P = ((p_{ij}))_{i,j \in I}\) is the transition matrix. In particular, using the fact that
\[p_{ij}^{(k)} := (P^k)_{ij} = \sum_{x_1, x_2, \ldots, x_{k-1} \in I} p_{ix_1} p_{x_1 x_2} \cdots p_{x_{k-1} j}\]
and summing over we have
\[
P(Y_i = y_i, i = 1, 2, \ldots, n) = P(X_{ki} = y_i, i = 0, 1, \ldots, n) = \lambda_{y_0} \prod_{i=1}^{n} p_{y_{i-1} y_i},
\]
for all \(n \geq 1, y_0, y_1, \ldots, y_n \in I\). Thus we have, \((Y_n)_{n \geq 0}^\infty\) is Markov\((\lambda, P^k)\).

2. (3 pts.) Show that for any \(i, j \in I, A \subseteq I\) and \(i \notin A\) where \(I\) is the state space, we have
\[
P_i(H_A^< \infty \mid X_1 = j) = P_j(H_A^< \infty)
\]
and
\[
E_i(H_A^< \mid X_1 = j) = 1 + E_j(H_A).
\]

**Solution:** Fix \(A \subseteq I, i, j \in I, i \notin A\). We have
\[
H_A := \inf\{n \geq 0 \mid X_n \in A\}
\]
Thus, when \(X_0 = i \notin A\), we have
\[
H_A = \inf\{n \geq 1 \mid X_n \in A\} = 1 + \inf\{n \geq 0 \mid X_{n+1} \in A\}.
\]
Hence,
\[
P_i(H_A^< \infty \mid X_1 = j) = P(\inf\{n \geq 0 \mid X_{n+1} \in A\} < \infty \mid X_1 = j, X_0 = i)
\]
\[
= P(\inf\{n \geq 0 \mid X_n \in A\} < \infty \mid X_0 = j)
\]
\[
= P_j(H_A^< \infty)
\]
where in the second equality we used Markov property and the fact that \(\inf\{n \geq 0 \mid X_{n+1} \in A\}\) depends only on \(X_k, k \geq 1\). Taking expectations we also get
\[
E_i(H_A^< \mid X_1 = j) = E(1 + \inf\{n \geq 0 \mid X_{n+1} \in A\} \mid X_1 = j, X_0 = i)
\]
\[
= 1 + E(\inf\{n \geq 0 \mid X_n \in A\} \mid X_0 = j) = 1 + E_j(H_A).
\]
3. (1+2+1+3+2 pts.) We consider a Markov chain which is given by a unidirectional ring with one escape point, *i.e.*, pick an integer $N > 1$ and $0 \leq p \leq 1$, and consider the Markov chain with state space $I = \{1, 2, \ldots, N+1\}$ and transition probabilities

\[
\begin{align*}
& p_{1,1} = 0, \quad p_{1,2} = p, \quad p_{1,N+1} = 1 - p, \\
& p_{i,i+1} = p, \quad p_{i,i} = 1 - p, \text{ for } i = 2, 3, \ldots, N - 1 \\
& p_{N,1} = p, \quad p_{N,N} = 1 - p, \\
& p_{N+1,N+1} = 1.
\end{align*}
\]

For $N = 4$ the corresponding graph is shown above. Prove the following:

(a) $N + 1$ is an absorbing state.

(b) If $0 < p < 1$, then $h_i^A(p) = 1$ for all $i = 1, 2, \ldots, N + 1$.

(c) If $p = 0$, the $h_i^A(p) = 0$ for all $i = 2, \ldots, N$, and $h_1^A(p) = h_{N+1}^A(p) = 1$.

(d) Compute $k_i^A(p)$ for all $i$ and all $0 \leq p \leq 1$. For which $i$ is this lowest and highest? Does this make sense?

(e) Show that $k_i^A(\cdot)$ is discontinuous at 0, *i.e.*, $k_i^A(0) \neq \lim_{p \to 0} k_i^A(p)$.

**Solution:**

(a) Clearly, $\{N + 1\}$ is a communicating class and it is closed.

(b) For simplicity we will not write the superscript $A$, unless needed. Fix $p \in (0, 1)$ and $i \in \{2, 3, \ldots, N - 1\}$. Then we have

\[
h_i = ph_{i+1} + (1 - p)h_i
\]

or

\[
ph_i = ph_{i+1}.
\]

Since $p > 0$, this means that $h_i = h_{i+1}$, so that $h_2 = h_3 = \cdots = h_N$. We also have

\[
h_N = ph_1 + (1 - p)h_N \text{ or } ph_1 = ph_N \text{ or } h_1 = h_N.
\]

Finally, note that

\[
h_1 = ph_2 + (1 - p)h_{N+1} = ph_2 + (1 - p)
\]
as $h_{N+1} = 1$. Using the fact that $h_1 = h_2$ and $p < 1$, we obtain $h_1 = 1$, and therefore $h_i = 1$ for all $i = 1, 2, \ldots, N$. 

(c) If $p = 0$, we can solve the Markov chain exactly. If $X_0 = N + 1$, then $X_n = N + 1$ for all $n \geq 1$. If $X_0 = 1$, then $X_n = N + 1$ for all $n \geq 1$. If $X_0 = i$ for some $i \in \{2, 3, \ldots, N\}$, then $X_n = i$ for all $n \geq 1$. The result follows.

(d) We approach these equations in a similar manner. Clearly, $k_{N+1} = 0$ for all $p$.

For $p = 0$, it is easy to see from the previous part that $k_1 = 1$, $k_2 = k_3 = \ldots = k_N = \infty$, $k_{N+1} = 0$.

For $p = 1$, we have $k_1 = k_2 = k_3 = \ldots = k_N = \infty$, $k_{N+1} = 0$.

Thus fix $p \in (0, 1)$. For $i = 2, 3, \ldots, N - 1$, we have

$$k_i = pk_{i+1} + (1-p)k_i + 1,$$

or $pk_i = pk_{i+1} + 1$,

or $k_i = k_{i+1} + \frac{1}{p}$.

We obtain a similar equation for $k_N$ and $k_1$, so we have $k_N = k_1 + \frac{1}{p}$ as well. We can thus recursively determine that

$$k_2 = k_N + \frac{N-2}{p} = k_1 + \frac{N-1}{p}.$$  

We now have that

$$k_1 = pk_2 + (1-p)k_{N+1} + 1,$$

or $k_1 = pk_2 + 1 = pk_1 + (N-1) + 1 = pk_1 + N$,

or $(1-p)k_1 = N$.

Thus when $0 < p < 1$, we have

$$k_1 = \frac{N}{1-p}, \quad k_i = k_1 + \frac{N + 1 - i}{p} \text{ for } i = 2, 3, \ldots, N.$$  

We see from these formulas that $k_1$ is the lowest (except of course for $k_{N+1} = 0$), and this makes sense, because all paths to $N + 1$ lead through state 1. Similarly, we see that $k_2$ is the highest, and again this makes sense from the topology: once we are at state 2, then we must go through the entire sequence of states to wrap around and get back to 1.

(e) For $p \in (0, 1)$ we have

$$k_i^A(p) = \frac{N}{1-p}, \text{ thus } \lim_{p \to 0^+} k_i^A(p) = N,$$

where as $k_1^A(0) = 1$. Thus $k_i^A(\cdot)$ is discontinuous at 0. This can be explained the discontinuity in the Markov chain at $p = 0$. For $p > 0$, $1 \rightarrow i \rightarrow 1$ for every $i = 2, 3, \ldots, N$, whereas for $p = 0$ we have $1 \not\rightarrow i \not\rightarrow 1$ for every $i = 2, 3, \ldots, N$ and $1 \rightarrow N + 1$ in one step with probability 1. When $p > 0$, starting from 1, the chain will come back to 1 in finite steps (with mean $1 + (N - 1)/p$) as the time spent at state $i$ is geometric with mean $1/p$ for $i = 2, 3, \ldots, N$ and 1 step to go from $N$ to 1) without visiting $N + 1$ with probability $p > 0$. Thus number of loops at 1 before visiting $N + 1$ is geometric with mean $p/(1 - p)$. The total amount of time spent at the loop $1 \rightarrow 1$ has mean $(N - 1 + p)/p \cdot p/(1 - p) = (N - 1 + p)/(1 - p) = N/(1 - p) - 1$. This explains the formula for $k_i^A(p)$ for $p \in (0, 1)$. For $p = 0$, the loops are all missing and it creates the discontinuity.

4. (1+2+1 pts.) Consider the transition matrix

$$P = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Draw the graph corresponding to the Markov chain, compute $P_i(X_n = j)$ for all $i, j \in \{1, 2, 3, 4\}$ and $n \geq 0$. Describe in words what happens to the stochastic process over long time.

**Solution:** Draw the Picture!!
It is easily to check that

\[
P^2 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}
\]

and \(P^3 = P\). Thus \(P^{2n+1} = P\) and \(P^{2n} = P^2\) for all \(n \geq 1\). Hence the chain oscillates between the two classes \(\{1, 3\}\) and \(\{2, 4\}\) with \(P_i(X_n = j) = 1/2\) if \(n > 0\) is even and \(i, j\) in the same class; or \(n\) is odd and \(i, j\) in different classes; and is zero otherwise.

5. (2+1 pts.) Let \(i, j, k \in I\) and \(m, n \geq 0\). Show that

\[p_{ij}^{(m+n)} \geq p_{ik}^{(m)} p_{kj}^{(n)}\]

where

\[p_{ij}^{(n)} = (P^n)_{ij}\]

Under what conditions are they equal?

**Solution:** Fix \(i, j, k \in I\) and \(m, n \geq 0\). We have

\[p_{ab}^{(t)} = \sum_{\pi = (a, x_1, x_2, \ldots, x_{t-1}, b)} p_{ax_1} p_{x_1x_2} \cdots p_{x_{t-1}b}\]

for all \(t \geq 0, a, b \in I\) where the sum is over all paths \(\pi\) of length \(t\) from \(a\) to \(b\). Clearly all terms inside the sum is non-negative. Thus restricting to paths from \(i\) to \(j\) of length \(m + n\) that goes through \(k\) at step \(m\) we get

\[p_{ij}^{(m+n)} \geq \sum_{\pi = (i, x_1, x_2, \ldots, x_{m-1}, k, \pi' = (k, y_1, y_2, \ldots, y_{n-1}, j)} p_{ix_1} p_{x_1x_2} \cdots p_{x_{m-1}k} p_{ky_1} p_{y_1y_2} \cdots p_{y_{n-1}j} = p_{ik}^{(m)} p_{kj}^{(n)}\]

Clearly, equality holds if all paths from \(i\) to \(j\) all length \(n + m\) goes through \(k\) at step \(m\), i.e.,

\[P(X_m = k \mid X_0 = i, X_{n+m} = j) = 1\]

6. (2+1 pts.) Show that every transition matrix on a finite state-space has at least one closed communicating class. Find an example of a transition matrix with no closed communicating class.

**Solution:** Recall that a communicating class \(C\) is closed if \(i \rightarrow j, i \in C\) implies \(j \in C\). Moreover, if \(i \rightarrow j\) and \(i, k\) are in the same communicating class, then \(k \rightarrow j\).

Now the state space is finite, hence there are finitely many communicating classes, say \(C_1, C_2, \ldots, C_n\). Assume that none of them are closed. Let \(t_1 = 1\). Since \(C_{t_1} = C_1\) is not closed, there exists \(i_1 \in C_{t_1}\) and \(i_2 \in C_{t_2}, t_2 \neq 1\), such that \(i_1 \rightarrow i_2\) (that is, the chain escapes from \(C_{t_1}\) to some other class \(C_{t_2}\)). Since the communicating class \(C_{t_2}\) is not closed, there exists \(i_3 \in C_{t_3}, t_3 \neq t_2\), such that \(i_2 \rightarrow i_3\). By continuing this process we obtain a sequence \((C_{t_k})_{k \geq 1}\) of communicating classes and a sequences of states, \((i_k)_{k \geq 1}\) such that \(i_k \in C_{t_k}\) and \(i_k \rightarrow i_{k+1}\) for all \(k \geq 1\). Since the number of communicating classes is finite there must be at least one class which appears at least twice in the sequence. Without loss of generality we assume that the class \(C_1\) appears twice. Thus we obtain that

\[i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_m \in C_1\]

for some \(m > 2\). Since \(i_m \rightarrow i_1\) (they are both in the same communicating class), we close the circle. This means that all states in the above sequence communicate. This is a contradiction with the fact that at least \(C_{t_2}\) is different from \(C_1\)!!

For the example, consider the Markov Chain with state space \(I = \{1, 2, \ldots\}\) with transition matrix \(P = ((p_{ij}))_{i,j \geq 1}\) where \(p_{i,i+1} = 1\) and 0 otherwise. Clearly, there are no closed communicating classes and \(i \rightarrow i + 1 \not\rightarrow i\) for all states \(i\).