26.1 Recap

**Discrete Time Branching Process** with $Z_0 = 1$: each individual will reproduce offspring $X \sim F$,

i. **Subcritical**: $E X < 1$, Extinction probability = 1;

ii. **Critical**: $E X = 0$, Extinction probability = 1;

iii. **Supercritical**: $E X > 1$, Extinction probability < 1;

**Markovian Branching Process (Continuous Time)** with death rate $\mu$ and birth rate $\lambda$,

i. **Subcritical**: $\lambda < \mu$, Extinction probability = 1;

ii. **Critical**: $\lambda = \mu$, Extinction probability = 1;

iii. **Supercritical**: $\lambda > \mu$, Extinction probability = $\mu/\lambda$ < 1;

26.2 Epidemic/Information Spreading/Flow of Liquid

Let $G = (V, E)$ denote the graph of individuals in a population. Let $I$ be the set of all the connected subgraphs of $G$. It is assumed that all neighboring pairs of individuals contact randomly and independently at a typical rate. The infection-spreading mechanism works so that individuals become infected when they contact an infected person. An infected person remains infected for a specific time and either recovers or dies. The infection-spreading and recovery mechanisms can be quantitatively explained using random variables to describe these mechanisms.

**Infection Spreading and Recovery Mechanism.** For the infection spreading mechanism, it can be assumed that each infected individual has an i.i.d. poisson clock with a rate parameter $\lambda$, and the infection is passed to a random neighbor when the Poisson clock ticks. The recovery mechanism can be modeled by its own Poisson clock with a rate parameter $\mu$, and the individual either recovers or dies when the clock ticks.

26.2.1 SIR (Susceptible/Infected/Recovered) Model

Suppose that there are $N$ individuals in a population. Let $S_0$ denote the number of susceptible individuals and $I_0$ denote the number of infected individuals at Time $t = 0$. At Time 0, each individual is susceptible or infected, $S_0 + I_0 = N$. If the state at a point of Time is $(s, i)$, then the model assumes that the number of susceptible and infected $X_t = (S_t, I_t)$ follows a CTMC. The state space for the model can be written as $I = \{(s, N - s) \mid 0 \leq s \leq N\}$. It is evident that the state $(N, 0)$ is absorbing, and all the other states are transient. For any state $X_t = (S_t, I_t) = (s, i)$,
the next state is either \( X_{t+1} = (s - 1, i + 1) \), a susceptible gets infected or \( X_{t+1} = (s, i - 1) \), an infected gets recovered. The transition rates are
\[
q_{(s,i),(s-1,i+1)} = \frac{\lambda \cdot i \cdot s}{N - 1}, \quad q_{(s,i),(s,i-1)} = \mu \cdot i, \quad i \geq 0.
\] (26.1)

### 26.2.2 Example: Rumor Spreading (SIR: \( \lambda = 1, \mu = 0, I_0 = 1, S_0 = N - 1 \))

The chain is \((N - 1, 1) \to (N - 2, 2) \to \cdots \to (0, N)\). Denote \( T \) as the Time for the rumor to spread to everyone. Then,
\[
T = t_1 + t_2 + t_3 + \cdots + t_{N-1},
\]
where \( t_i \) are independent of each other and \( t_i \sim \text{Exp} \left( \frac{i(N-i)}{N-1} \right) \). We have
\[
\mathbb{E} T = \sum_{i=1}^{N-1} \mathbb{E} t_i = \sum_{i=1}^{N-1} \frac{N - 1}{i(N - i)} = N - 1 - \sum_{i=1}^{N-1} \left( \frac{1}{N - k} + \frac{1}{k} \right) = 2 \cdot \frac{N - 1}{N} \cdot \sum_{i=1}^{N-1} \frac{1}{k} \approx 2 \log N.
\]

**Theorem 26.1.** We have, \( \frac{\mathbb{E} T}{2 \log N} \to 1 \) and \( \frac{T}{2 \log N} \to 1 \) in probability as \( N \to \infty \).

**Theorem 26.2.** Consider the Complete Graph on \( N \) vertices. Assume that rumor starts from vertex \( i \). Define \( T_j := \text{Time when the rumor arrives at vertex } j \). Then
\[
\frac{T_j}{\log N} \to 1 \text{ a.s. as } N \to \infty
\]

**Idea:** Given \( I_0 = 1 \) starting with vertex \( i \) at time 0, we have \( T_1 := \inf \{ t \mid |I_t| \geq \sqrt{N} \} \sim \sum_{k=1}^{\sqrt{N}} \text{Exponential} \left( \frac{k(N-k)}{N-1} \right) \) with \( \mathbb{E} T_1 = \frac{N-1}{N} \cdot \sum_{k=1}^{\sqrt{N}} \left( \frac{1}{N-k} + \frac{1}{k} \right) \approx \frac{1}{2} \log N. \) Same also holds around vertex \( j \).

### 26.3 Queuing Model (M/M/s)

Queues form in many circumstances, and predicting their behavior is essential for the system’s efficiency. The basic mathematical model for queues assumes that a number of customers are

\[ \sqrt{\frac{N}{N-1}} = 1 \]

\[ \frac{1}{2} \log(N) \]
waiting for service, and each customer has to wait for some time before he/she gets served. The
arrival times and waiting times are independently distributed variables.

\( M \): Memoryless inter-arrival times \( \sim \) Exponential\((\lambda)\)

\( M \): Memoryless service times \( \sim \) Exponential\((\mu)\)

\( s \): Number of servers, \( 1 \leq s \leq \infty \).

### 26.3.1 M/M/1

M/M/1 queue is the simplest queuing network model. M/M/1 means memoryless inter-arrival times/memoryless service times/one server. Suppose that the inter-arrival times are exponentially distributed with parameter \( \lambda \) and the service times are exponential of parameter \( \mu \). Then the number of customers in the queue \((X_t)_{t \geq 0}\) evolves as a Markov chain, and the state transition diagram is given below. Here the State space \( I \) is \( \{0, 1, 2, \ldots \} \).

![](image)

The \( Q \) matrix is

\[
Q = \begin{bmatrix}
-\lambda & \lambda & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
\mu & -(\lambda + \mu) & \lambda & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & \mu & -(\lambda + \mu) & \lambda & \ldots & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & \ldots & \mu & -(\lambda + \mu) & \lambda & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}
\]

The jump chain corresponding to the \( Q \) matrix is a random walk with probability

\[
\Pi_{0,1} = 1, \quad \Pi_{i,i+1} = \frac{\lambda}{\lambda + \mu}, \quad \Pi_{i,i-1} = \frac{\mu}{\lambda + \mu} \quad \text{for } i \in I,
\]

with

\[
\Pi = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
\frac{\mu}{\lambda + \mu} & 0 & \frac{\lambda}{\lambda + \mu} & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & \frac{\mu}{\lambda + \mu} & 0 & \frac{\lambda}{\lambda + \mu} & \ldots & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & \ldots & \frac{\mu}{\lambda + \mu} & 0 & \frac{\lambda}{\lambda + \mu} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}
\]

When \( \lambda > \mu \), \( X_t \) is transient and \( X_t \to \infty \). When \( \lambda < \mu \), \( X_t \) is positive recurrent. Under condition \( \lambda < \mu \), the equilibrium distribution is

\[
\pi_i = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^i, \quad i > 0.
\]
The average number of customers in the queue in equilibrium is given by

$$E_\pi(X_t) = \sum_{i=1}^{\infty} P_\pi(X_t \geq i) = \frac{\lambda}{\mu - \lambda}.$$

### 26.3.2 M/M/s

This queueing model is similar to M/M/1 queue, but instead of 1 server, there are $s$ servers, and the service rate by each server is $\mu$. Thus, the first service time is exponential of parameter $\mu$, and the maximum service rate occurs when all the servers are working and is $s\mu$. Therefore, the queue size is a Markov chain $(X_t)_{t \geq 0}$ with the state transition diagram below. We have the rate matrix

$$Q = \begin{bmatrix}
-\lambda & \lambda & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
\mu & -(\lambda + \mu) & \lambda & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 2\mu & -(\lambda + 2\mu) & \lambda & \ldots & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & \ldots & s\mu & -(\lambda + s\mu) & \lambda & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}.$$

The jump chain corresponding to $Q$ matrix is a random walk with probability

$$\Pi_{01} = 1, \Pi_{i(i+1)} = \frac{\lambda}{\lambda + i\mu}, \Pi_{i(i-1)} = \frac{i\mu}{\lambda + i\mu} \text{ for } i \in I,$$

and

$$\Pi = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
\frac{\mu}{\lambda + \mu} & 0 & \frac{\lambda}{\lambda + \mu} & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & \frac{2\mu}{\lambda + 2\mu} & 0 & \frac{\lambda}{\lambda + 2\mu} & \ldots & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & \ldots & \frac{s\mu}{\lambda + s\mu} & 0 & \frac{\lambda}{\lambda + s\mu} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}.$$

The chain is transient when $\lambda \geq s\mu$ and is recurrent otherwise. Application of M/M/s model includes but is not limited to,

i. Genetic Divergence: Wright-Fisher model and Moran model,

ii. Insurance Claims: Bankruptcy model.