23.1 Time Reversal

**Theorem 23.1.** Let $Q$ be an irreducible non-explosive rate matrix with invariant distribution $\lambda$. Fix $T \in (0, \infty)$. Define, $\tilde{X}_t = X_{T-t}$, $0 \leq t \leq T$. Then,

$$(\tilde{X}_t)_{0 \leq t \leq T} \sim \text{Markov}(\lambda, \tilde{Q})$$

where $\lambda_i \tilde{q}_{ij} = \lambda_j q_{ji}$ for all $i, j$. Moreover, $\tilde{Q} = ((\tilde{q}_{ij}))$ is irreducible, non-explosive with invariant distribution $\lambda$.

**Proof.** The transition matrix for $(X_t)_{t \geq 0}$ satisfies the forward equation $P'(t) = P(t)Q, t > 0$. Define $\tilde{q}_{ij} = \lambda_j q_{ji}/\lambda_i$, $\tilde{p}_{ij}(t) = \lambda_j p_{ji}(t)/\lambda_i, i, j \in I, t > 0$ and $\tilde{P}(t) = ((\tilde{p}_{ij}(t)))$.

We claim that

i. $\tilde{Q}$ is a rate matrix, and

ii. $\tilde{P}'(t) = \tilde{Q}\tilde{P}(t)$ for all $t > 0$.

The proof of i. is easy. For ii., fix $i, j$. We have

$$\tilde{p}_{ij}(t) = \left(\frac{\lambda_j p_{ji}(t)}{\lambda_i}\right)' = \frac{\lambda_j}{\lambda_i} (P(t)Q)_{ij} = \frac{\lambda_j}{\lambda_i} \sum_k p_{jk}(t)q_{ki}$$

$$= \sum_k \frac{\lambda_j}{\lambda_i} p_{jk}(t) \frac{\lambda_k}{\lambda_i} q_{ki} = \sum_k \tilde{p}_{jk}(t) \tilde{q}_{ik} = (\tilde{Q}\tilde{P}(t))_{ij}.$$

Trivially, $\tilde{P}(0) = I$. Thus $\tilde{P}(t), t \geq 0$ satisfies the backwards equations with rate matrix $\tilde{Q}$.

**Exercise 23.1.** Check that $\tilde{Q}$ is non-explosive, irreducible with invariant distribution $\lambda$.

Now, fix times $0 \leq t_1 < t_2 < \cdots < t_k = T$. We want to show that

$$P(\tilde{X}_{t_i} = x_i, i = 1, 2, \ldots, k) = \lambda_{x_1} \tilde{p}_{x_1 x_2}(t_2 - t_1)\tilde{p}_{x_2 x_3}(t_3 - t_2)\cdots\tilde{p}_{x_{k-1} x_k}(t_k - t_{k-1}).$$

We have the LHS equal to

$$P(X_{T-t_i} = x_i, i = 1, 2, \ldots, k) = P(X_{T-t_k} = x_k, \ldots, X_{T-t_1} = x_1)$$

$$= \lambda_{x_k} p_{x_k x_{k-1}}(t_k - t_{k-1})\cdots p_{x_2 x_1}(t_2 - t_1)$$

$$= \frac{\lambda_{x_k}}{\lambda_{x_{k-1}}} p_{x_k x_{k-1}}(t_k - t_{k-1}) \cdot \frac{\lambda_{x_{k-1}}}{\lambda_{x_{k-2}}} p_{x_{k-1} x_{k-2}}(t_{k-1} - t_{k-2})\cdots \frac{\lambda_{x_2}}{\lambda_{x_1}} p_{x_2 x_1}(t_2 - t_1) \lambda_{x_1}$$

$$= \lambda_{x_1} \tilde{p}_{x_1 x_2}(t_2 - t_1)\cdots\tilde{p}_{x_{k-1} x_k}(t_k - t_{k-1}) = \text{RHS}.$$
This completes the proof.

### 23.2 Detailed Balance

A distribution \( \lambda \) is in detailed balance with a rate matrix \( Q \) iff

\[
\lambda_i q_{ij} = \lambda_j q_{ji}, \forall i \neq j.
\]

**Theorem 23.2.** Assume that \( Q \) is irreducible, non-explosive with invariant distribution \( \lambda \). The following are equivalent

i. \( \lambda \) is in detailed balance with \( Q \),

ii. The CTMC is time reversible.

**Corollary 23.3.** \( \lambda \) is in detailed balance with \( Q \) implies that \( \lambda \) is invariant for \( Q \).

Note that a distribution \( \lambda \) is in detailed balance with \( Q \) implies that \( \lambda \) is invariant for \( Q \), but \( Q \) may be explosive.

**Example 23.4.** Consider the state space \( I = \mathbb{Z}_+ \) and the birth and death CTMC with \( q_{i,i+1} = p \cdot q_i, q_{i,i-1} = q \cdot q_i, i \geq 0 \) where \( p + q = 1 \). Assume that \( p > q \). The jump chain is

\[
\Pi_{ij} = \begin{cases} 
p & \text{if } j = i + 1, \\
q & \text{if } j = i - 1, \\
0 & \text{otherwise.}
\end{cases}
\]

If \( p = q \), then the jump chain is recurrent. If \( p > q \), then the chain goes to \( \infty \). Thus, \( (Y_n)_{n \geq 0} \) is transient. Let us find \( \lambda \) such that \( \lambda_i q_{ij} = \lambda_j q_{ji} \) for \( i \neq j \). It is enough to consider \( j = i + 1 \) and solve

\[
\lambda_i q_i \cdot p = \lambda_{i+1} q_{i+1} \cdot q, i \geq 0.
\]

Solving we get,

\[
\lambda_{i+1} q_{i+1} = \lambda_i q_i \cdot \frac{p}{q} = \lambda_0 q_0 \cdot \left( \frac{p}{q} \right)^{i+1}.
\]

Thus,

\[
\lambda_i \propto \frac{1}{q_i} \left( \frac{p}{q} \right)^i \text{ for all } i \geq 0.
\]

If \( q_i = (2p/q)^i \), then \( \lambda_i \propto \frac{1}{q_i} \), \( i \geq 0 \) implies that there exists an invariant measure.

**Remark 23.5.** Recurrence \( \implies \) Non-explosive; but the reverse is not true. Non-explosive + Invariance \( \implies \) Recurrence.

### 23.3 Convergence to Equilibrium

Let \( Q \) be irreducible and non-explosive with invariant distribution \( \lambda_i = 1/m_i q_i, i \in I \).
Theorem 23.6. Let \((X_t)_{t \geq 0} \sim \text{Markov}(\nu, Q)\). Then, \(\mathbb{P}(X_t = j) \xrightarrow{t \to \infty} \lambda_j \) for all \(j \in I\).

Equivalently, \(d_{TV}(\nu^t P(t), \nu^\infty) = \frac{1}{2} \sum_{j \in I} |\mathbb{P}(X_t = j) - \lambda_j| \xrightarrow{t \to \infty} 0\). (Exercise!)

Proof. Fix a positive real number \(h > 0\). Define \(Z_n = X_{nh}, n \geq 0\). Thus, \((Z_n)_{n \geq 0} \sim \text{Markov}(\nu, P(h))\).

Now, \(\lambda\) is invariant for \(Q\), i.e., \(\lambda^\top Q = 0\) and \(Q\) is non-explosive, implies that \(\lambda^\top P(h) = \lambda^\top\). Thus, \(P(h)\) is irreducible, aperiodic, positive recurrent with invariant distribution \(\lambda\). In particular, \(\mathbb{P}(Z_n = j) = \mathbb{P}(X_{nh} = j) \xrightarrow{n \to \infty} \lambda_j \forall j \in I\).

Fix \(t > 0\). There exists \(n \in \mathbb{N}\) such that \(nh \leq t < (n + 1)h\). We want to bound

\[
\sup_n \sup_{nh \leq t < (n + 1)h} |\mathbb{P}(X_t = j) - \mathbb{P}(X_{nh} = j)|.
\]

Lemma 23.7. Take \(s \leq t < s + h\). Then \(|\mathbb{P}(X_t = j) - \mathbb{P}(X_s = j)| \leq 1 - e^{-q_i h}\).

Proof of Lemma 23.7. Clearly,

\[
\mathbb{P}(X_t = j) - \mathbb{P}(X_s = j) = \sum_k \mathbb{P}(X_t = j, X_s = k) - \mathbb{P}(X_s = j) \\
= \sum_k \mathbb{P}(X_s = k) \mathbb{P}(X_{t-s} = j) - \mathbb{P}(X_s = j) \\
= -\mathbb{P}(X_s = j)(1 - \mathbb{P}(X_{t-s} = i)) + \sum_{k \neq i} \mathbb{P}(X_{t-s} = k) \mathbb{P}(X_s = j).
\]

Thus, \(|\mathbb{P}(X_t = j) - \mathbb{P}(X_s = j)| \leq 1 - p_{ii}(t - s) \leq 1 - e^{-q_i(t-s)} \leq 1 - e^{-q_i h}.\)

Using the Lemma, we get that

\[
\sup_n \sup_{nh \leq t < (n + 1)h} |\mathbb{P}(X_t = j) - \mathbb{P}(X_{nh} = j)| \leq \sum_i \nu_i(1 - e^{-q_i h}) \xrightarrow{h \to 0} 0.
\]

Thus, \(\mathbb{P}(X_t = j) \to \lambda_j\) as \(t \to \infty\) for all \(j \in I\).

We now state the Ergodic theorem for irreducible, non-explosive, positive recurrent CTMC.

Theorem 23.8. Assume that \(Q\) is irreducible, non-explosive with invariant distribution \(\lambda\) with \(\lambda_j = \frac{1}{m_j q_i}, j \in I\). We have, almost surely,

\[
\frac{1}{t} \int_0^t \mathbb{1}_{X_s = j} ds \xrightarrow{t \to \infty} \lambda_j \text{ for all } j \in I,
\]
and for any bounded function \( f \) on \( I \),
\[
\hat{f}_t = \frac{1}{t} \int_0^t f(X_s) \, ds \xrightarrow{t \to \infty} \sum_{i \in I} f_i \lambda_i.
\]

**Remark 23.9.** One can prove that \( \sqrt{t} \left( \hat{f}_t - \sum_{i \in I} f_i \lambda_i \right) \xrightarrow{\text{in distribution}} N(0, \sigma^2) \) in distribution under appropriate conditions, as \( t \to \infty \) for some \( \sigma \geq 0 \).