14.1 Discrete Time and Discrete Space Markov Chain

14.1.1 Structure Decomposition

Consider a Markov Chain \((X_t)_{t \geq 0} \sim \text{Markov}(\lambda, P)\) on a countable state space \(I\) where \(\lambda = (\lambda_i)_{i \in I}\) is the initial distribution and \(P = (p_{ij})_{i,j \in I}\) is the transition matrix.

By our previous result, we can decompose the state space \(I\) into disjoint union of Communication Classes (CCs), i.e., \(I = C_1 \cup C_2 \cup \cdots \cup C_n \cup D_1 \cup \cdots\) where \(C_i\)'s are closed communicating classes and \(D_i\)'s are not closed communicating class.

Also, for each \(C_i\), take \(C_1\) as an example here; with its period \(d\), it has the following structure:
14.2 Time Reversal

**Theorem 14.1.** Let $P$ be irreducible and have an invariant distribution $\pi$. Suppose a Markov Chain $(X_t)_{t \geq 0} \sim \text{Markov}(\pi, P)$. Take $N \geq 1$ and set $Y_n = X_{n-n}$, for $n = 0, 1, \ldots, n$. Then, we have another Markov Chain $(Y_t)_{t=0}^{N} \sim \text{Markov}(\pi, \hat{P})$, where $\hat{P}_{ij} = \frac{\pi_i P_{ij}}{\pi_j}$, $\forall i, j \in I$. Moreover, $\pi$ is invariant distribution for $\hat{P}$.

**Remark 14.2.**

\[
\pi_i \hat{P}_{ij} = \pi_j P_{ij}
\]

\[
\text{LHS} = P(Y_0 = i) P(Y_1 = j|Y_0 = i) = P(Y_0 = i, Y_1 = j)
\]

\[
\text{RHS} = P(X_0 = j) P(X_1 = i|X_0 = j) = P(X_0 = j, X_1 = i)
\]

**Proof.** Let us check the following:

- First, we check that $\hat{P}$ is a transition matrix. We have $\hat{P}_{ij} \geq 0$, for all $i, j$ and $\sum_{j \in I} \hat{P}_{ij} = \sum_{j \in I} \pi_j P_{ji} / \pi_i = \pi_i / \pi_i = 1$ since $\pi$ is invariant for $P$.

- Next, we check that $\pi$ is a stationary distribution for $\hat{P}$, which is to show $\sum_{i \in I} \pi_i \hat{P}_{ij} = \pi_j$ for all $i$. We have, $LHS = \sum_{i \in I} \pi_i \frac{\pi_j P_{ji}}{\pi_i} = \pi_j = RHS$ since $P$ is stochastic matrix. Therefore we have shown that $\pi$ is an invariant distribution for $\hat{P}$.

- Finally we want to show $(Y_n)_{n \geq 0}$ is Markov$(\pi, \hat{P})$, which is equivalent to show that

\[
P(Y_0 = i_0, \ldots, Y_i = i_N) = \pi_{i_0} \hat{P}_{i_0,j_0} = \pi_{i_0} \hat{P}_{i_1,j_1} \cdots \hat{P}_{i_{n-1},j_{n-1},i_1} \cdots \hat{P}_{i_N,j_N}, \forall i_1, \ldots, i_N.
\]

We have,

\[
P(Y_0 = i_0, \ldots, Y_N = i_N) = P(X_N = i_N, \ldots, X_1 = i_1)
\]

\[
= \pi_{i_N} P_{i_N,i_{N-1}} \cdots P_{i_1,i_0} = \pi_{N-1} \hat{P}_{i_{N-1},i_N} \pi_{N-1} \hat{P}_{i_{N-2},i_{N-1}} \cdots P_{i_1,i_0}
\]

(BY Induction)

\[
= \hat{P}_{i_{N-1},i_N} \hat{P}_{i_{N-2},i_{N-1}} \cdots \pi_{i_0} \hat{P}_{i_0,j_1},
\]

so $\hat{P}$ is also irreducible.

The chain $(Y_n)_{0 \leq n \leq N}$ is called the time-reversal of $(X_n)_{0 \leq n \leq N}$.

**Definition 14.3.** A Markov chain with transition matrix $P$ and invariant distribution $\pi > 0$ is called **reversible** if $\hat{P} = P$. In particular, $\pi_i P_{ij} = \pi_j P_{ji}$, for all $i, j \in I$.

**Definition 14.4.** We say that a distribution $\lambda$ and a transition matrix $P$ are in detailed balance if

\[
\lambda_i P_{ij} = \lambda_j P_{ji}, \forall i, j.
\]

Though obvious, the following result is worth remembering because, when a solution $\lambda$ to the detailed balance equations exists, it is often easier to find by the detailed balance equations than by the equation $\lambda = \lambda P$. 

Theorem 14.5. If \((\lambda, P)\) is in detailed balance, then \(\lambda\) is the invariant distribution for \(P\).

Proof. To show this, we want \(\sum_i \lambda_i P_{ij} = \lambda_j, \forall j\), which follows from \(\lambda_i P_{ij} = \lambda_j P_{ji}, \forall i, j\).

Theorem 14.6. Let \(P\) be an irreducible stochastic matrix, and let \(\lambda\) be a distribution. Suppose that \((X_n)_{n \leq 0}\) is Markov\((\lambda, P)\). Then the following are equivalent:

1. \((X_t)_{t \geq 0}\) is reversible.
2. \((\lambda, P)\) is in detailed balance.

14.2.1 Random Walk on Locally Finite Graph

For a undirected connected graph, \(G = (V, E)\), define \(d(v) = \text{degree of } v = \text{number of edges adjacent to } v\). Define the graph to be “finite” if it has finitely many vertices, and we have \(d(v) < \infty, \forall v \in V\).

The transition probability is given by: \(p_{uv} = \begin{cases} 0, & \text{if } (u, v) \notin E \\ \frac{1}{d(v)}, & \text{if } (u, v) \in E. \end{cases}\) Then, \(\lambda = (d(u)/2|E|)_{u \in V}\) is a probability measure.

Lemma 14.7. \((\lambda, P)\) is in detailed balance.

Proof. Want to show that \(\lambda_u P_{uv} = \lambda_v P_{vu}\), for all \(u, v\), and we have:

1. if \((u, v) \notin E\), both are 0.
2. if \((u, v) \in E\), then \(\lambda_u = \frac{d(u)}{2|E|}\). Since LHS = \(\frac{d(u)}{2|E|} \cdot \frac{1}{d(u)} = \frac{1}{2|E|}\), which is equivalent to RHS.

Corollary 14.8. For a random walk, If \(G\) is finite, \(\pi = \left(\frac{\lambda(u)}{2|E|}\right)_{u \in V}\) is in detailed balance with \(P\) and is the unique invariant distribution. The random walk is either null recurrent, or transient if \(G\) is infinite.

Theorem 14.9. For Random walk on \(\mathbb{Z}^d\). If \(d \leq 2\), then it is null recurrent; If \(d \geq 3\), then it is transient.

Proof. Enough to show that state 0 is null recurrent, or transient. We have,

\[
\sum_{n=0}^{\infty} P_{00}^n = \begin{cases} \infty, & \text{if recurrent} \\ \text{finite}, & \text{if transient}. \end{cases}
\]

1. For \(d = 1\), we have

\[
P_{00}^{2n} = \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} = \frac{2n!}{n!^{12} \cdot 2^{2n}}
\]
14-4 Lecture 14: Time Reversal and Detailed Balance

By Stirling’s Approximation: \( n! \approx \sqrt{2\pi e^{-n/2} n^{n+1/2}} \) for large \( n \), and thus

\[
P_{00}^{2n} = \frac{\sqrt{2\pi e^{-2n} (2n)^{2n+1/2}}}{\sqrt{2\pi e^{-2n} n^{2n+1/2}}} \approx \frac{1}{\sqrt{\pi n}}
\]

and

\[
\sum_{n=0}^\infty P_{00}^{2n} = \infty.
\]

Therefore, the random walk in the 1D case is recurrent.

2. For \( d = 2 \),

\[
P_{00}^{2n} = \sum_{k=0}^n \frac{(2n)!}{k!(n-k)!(n-k)!} \left( \frac{1}{4} \right)^{2n} = \frac{1}{2^{2n}} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2 \approx \frac{1}{\pi n}.
\]

Thus, \( \sum_{n=0}^\infty P_{00}^{2n} = \infty \). Note that, \( \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n} \), which can be proved from \( (1+x)^n(1+x)^n = (1+x)^{2n} \). Therefore, the random walk in the 2D case is recurrent.

3. For \( d = 3 \),

\[
P_{00}^{2n} = \sum_{k_1+k_2+k_3=n} \frac{(2n)!}{k_1!k_2!k_3!} \left( \frac{1}{2d} \right)^{2n} = \left( \frac{1}{2d} \right)^{2n} \binom{2n}{n} \sum_{k_1+k_2+k_3=n} \binom{n}{k_1,k_2,k_3}^2.
\]

Note: \( \binom{n}{k} \) is largest when \( \frac{n-1}{2} \leq k \leq \frac{n+1}{2} \), and \( \binom{n}{k_1,k_2,...,k_d} \) is largest when \( |k_i - k_j| \leq 1, \forall i, j \)
Then, for \( n = 3m \), the above equation can also be written as

\[
P_{00}^{6m} \leq \frac{1}{6^{6m}} \binom{6m}{3m} \sum_{k_1+k_2+k_3=3m} \binom{3m}{k_1,k_2,k_3}.
\]

Note that the summation is the \( 3^m \) multinomial expansion, and thus

\[
= \frac{1}{2^{2n}} \cdot \binom{2n}{n} \cdot \frac{1}{3^{3m}} \cdot \frac{(3m)!}{m!^3} \approx \frac{1}{C n^{3/2}}
\]

In general, \( d \geq 1, P_{00}^{2n} \leq \frac{C}{n^{d/2}} \). Therefore, \( \sum_{n=0}^\infty P_{00}^{2n} < \infty \) iff \( d > 2 \).

Exercise 14.1. If \( G \) is recurrent, then any connected subgraph is also recurrent.