11.1 Invariant distributions

We first prove the theorem from the last lecture.

**Theorem 11.1.** Let $P$ be irreducible and let $\lambda$ be an invariant measure for $P$ with $\lambda_k = 1$. Then $\lambda \geq \gamma^{(k)}$. If in addition $P$ is recurrent, then $\lambda = \gamma^{(k)}$.

**Proof.** For any $j \in I$ we have

$$\lambda_j = \sum_{i_1 \in I} \lambda_{i_1} p_{i_1 j} = \sum_{i_1 \neq k} \lambda_{i_1} p_{i_1 j} + p_k j = \sum_{i_1, i_2 \neq k} \lambda_{i_2} p_{i_2 i_1} p_{i_1 j} + p_{k_1} p_{i_1 j} + \sum_{i_1 \neq k} p_{k_1} p_{i_1 j}$$

$$\vdots$$

$$= \sum_{i_1, \ldots, i_n \neq k} \lambda_{i_n} p_{i_n i_{n-1}} \cdots p_{i_1 j} + \left( p_{k_j} + \sum_{i_1 \neq k} p_{k_1} p_{i_1 j} + \cdots + \sum_{i_1, \ldots, i_{n-1} \neq k} p_{k_{n-1}} \cdots p_{i_2 i_1} p_{i_1 j} \right)$$

Dropping the non-negative term $\sum_{i_1, \ldots, i_n \neq k} \lambda_{i_n} p_{i_n i_{n-1}} \cdots p_{i_1 j}$, we obtain

$$\lambda_j \geq P_k(X_1 = j, R_k \geq 1) + P_k(X_2 = j, R_k \geq 2) + \cdots + P_k(X_n = j, R_k \geq n) \to \gamma^{(k)}_j$$

as $n \to \infty$ where $R_k$ is the first return time to $k$. So, we have $\lambda \geq \gamma^{(k)}$.

If in addition $P$ is recurrent, then we proved earlier that $\gamma^{(k)}$ is invariant, so $\mu = \lambda - \gamma^{(k)}$ is also invariant with $\mu \geq 0$ and $\mu_k = \lambda_k - \gamma^{(k)} = 0$. Since $P$ is irreducible, for any $i \in I$, there is some $n$ such that $p_{i_k}^{(n)} > 0$.

Now, we have

$$0 = \mu_k = \sum_{j \in I} \mu_j p_{j_k}^{(n)} \geq \mu_i p_{i_k}^{(n)},$$

so $\mu_i = 0$, and $\lambda = \gamma^{(k)}$.

Recall that, $i \in I$ is recurrent $\iff P_i(X_n = i \text{ infinity often}) = 1 \iff P_i(R_i < \infty) = 1$. Let $m_i$ be the expected first return time

$$m_i = E_i(R_i).$$
**Definition 11.2 (Positive recurrent).** A state \(i\) is positive recurrent if \(m_i < \infty\), otherwise it is null recurrent.

**Theorem 11.3.** Let \(P\) be irreducible, then the following are equivalent:

i. every state is positive recurrent;

ii. some state is positive recurrent;

iii. \(P\) has an invariant distribution \(\pi\).

Moreover, when iii. holds, \(m_k = 1/\pi_k\) for all \(k \in I\).

*Proof.* (i) \(\implies\) (ii). This is trivial.

(ii) \(\implies\) (iii) If \(i\) is positive recurrent, then \(P\) is recurrent, \(\gamma(i)\) is invariant. We have

\[
\sum_{j \in I} \gamma_{ij} = m_i < \infty
\]

So \(\gamma(i)/m_i\) is an invariant distribution.

(iii) \(\implies\) (i) Take arbitrary \(k \in I\). Since \(P\) is irreducible and \(\sum_{i \in I} \pi_i = 1\), we have \(\pi_k = \sum_{i \in I} \pi_i P_{ik}^{(n)} > 0\) for some \(n\). Set \(\lambda_i = \pi_i/\pi_k\). Then \(\lambda\) is invariant and \(\lambda_k = 1\). By Theorem 11.1, \(\lambda \geq \gamma(k)\), so

\[
m_k = \sum_{i \in I} \gamma_{ik} \leq \sum_{i \in I} \frac{\pi_i}{\pi_k} = \frac{1}{\pi_k} < \infty
\]

and \(k\) is positive recurrent.

Finally, suppose that (iii) holds, then \(P\) is positive recurrent, then by theorem 11.1 the inequality \(\lambda \geq \gamma(k)\) is an equality. So \(m_k = 1/\pi_k\) for all \(k \in I\).

*Example 11.4 (Simple symmetric random walk on \(Z\)).* In the simple symmetric random walk problem, the state space is \(Z\), the probabilities of transitioning from state \(i\) to \(i + 1\) or \(i - 1\) are both 1/2. We know that \(P\) is recurrent. Consider

\[
\pi_i = 1 \text{ for all } i.
\]

Clearly, \(\pi\) is invariant. Then we can use Theorem 11.1 to see any invariant measure is a multiple of \(\pi\). So there is no invariant distribution.

*Example 11.5 (Simple symmetric random walk on \(Z^3\)).* Similarly, consider \(\pi_i = 1\) for all \(i\). Clearly \(\pi\) is an invariant measure. But we know that simple symmetric random walk on \(Z^3\) is transient. So the existence of invariant measure does not imply recurrence.

*Example 11.6 (Asymmetric random walk on \(Z\)).* Consider the asymmetric random walk on \(Z\) with \(p_{i,i-1} = q < p = p_{i,i+1}\). An invariant measure \(\pi\) needs to satisfy \(\pi^T = \pi^T P\), that is, for any \(i \in Z\),

\[
\pi_i = \pi_{i-1} p + \pi_{i+1} q
\]

We can check the solution is

\[
\pi_i = A + B \cdot (p/q)^i, \quad i \in Z.
\]
Example 11.7. Consider a Markov Chain on $\mathbb{Z}^+$ so that for all $i \geq 0$,

$$
\begin{align*}
p_{i,i+1} &= p_i \\
p_{i,0} &= q_i = 1 - p_i, \ i \in \mathbb{Z}^+.
\end{align*}
$$

The equation $\pi^\top = \pi^\top P$ gives

$$
\pi_0 = \sum_{i=0}^\infty q_i \pi_i
$$

$$
\pi_i = p_{i-1} \pi_{i-1}, \ \text{for all } i \geq 1
$$

Combining the equations, we can see,

$$
\pi_0 = \sum_{i=0}^\infty (1 - p_i) p_{i-1} \cdots p_0 \cdot \pi_0.
$$

Suppose we choose $p_i$ sufficiently quickly converging to 1 so that

$$
p = \prod_{i=0}^\infty p_i > 0.
$$

We notice

$$
\sum_{i=0}^\infty (1 - p_i) p_{i-1} \cdots p_0 = (1 - p_0) + (1 - p_1) p_0 + (1 - p_2) p_1 p_0 + \cdots.
$$

is a telescoping series, and

$$
\sum_{i=0}^\infty (1 - p_i) p_{i-1} \cdots p_0 = 1 - \prod_{i=0}^\infty p_i = 1 - p < 1.
$$

Hence,

$$
\pi_0 = (1 - p) \pi_0
$$

implies that $\pi_0 = 0$, and $\pi_i = 0$ for all $i$. So the only invariant measure is the trivial zero invariant measure.

11.2 Aperiodicity

Consider $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. we can see $P^2 = I, P^3 = P$ and more generally, $P^{2n} = I, P^{2n+1} = P$ for all integer $n \geq 0$. Clearly $p_{ij}^{(n)}$ does not converge for all $i, j$. In this case, the states are not aperiodic.

**Definition 11.8 (Aperiodic state).** State $i$ is aperiodic if $p_{ii}^{(n)} > 0$ for all sufficiently large $n$.

**Exercise 11.1.** State $i$ is aperiodic if and only if $\gcd \left\{ n : p_{ii}^{(n)} > 0 \right\} = 1$. 
Lemma 11.9. Suppose $P$ is irreducible with aperiodic state $i$. Then for all states $j$ and $k$, $p_{jk}^{(n)} > 0$ for all sufficiently large $n$. In particular, all states are aperiodic.

Proof. $P$ is irreducible implies that for some $r, s \geq 0$, $p_{ji}^{(r)} > 0$ and $p_{ik}^{(s)} > 0$. Then,

$$p_{jk}^{(r+n+s)} \geq p_{ji}^{(r)} p_{ii}^{(n)} p_{ik}^{(s)} > 0$$

for all sufficiently large $n$. ■

Theorem 11.10. Suppose $P$ is irreducible and aperiodic with invariant distribution $\pi$. Let $\lambda$ be any distribution. Suppose $(X_n)_{n \geq 0} \sim \text{Markov}(\lambda, P)$, then

$$P(X_n = j) \rightarrow \pi_j \text{ as } n \rightarrow \infty$$

for all $j$. In particular, $p_{ij}^{(n)} \rightarrow \pi_j$ as $n \rightarrow \infty$ for all $j$.

Proof. Next lecture. ■