This lecture defines the recurrence and transience of states in a Markov chain. We prove the dichotomy that either a state is transient or recurrent. We also show that all states are recurrent or transient in a communication class.

### 9.1 Recurrence and Transience

**Definition 9.1.** Let \((X_t)_{t \geq 0}\) be a Markov chain. A state \(i \in I\) is **recurrent** if
\[
P_i(X_t = i \text{ for infinitely many } t) = 1.
\]

A state \(i \in I\) is **transient** if
\[
P_i(X_t = i \text{ for infinitely many } t) = 0.
\]

In other words, a recurrent state is one where the MC keeps coming back, while a transient state is one where the chain eventually leaves and never returns. The following theorem shows that every state is either recurrent or transient.

**Theorem 9.2 (Dichotomy).** Any state is either **recurrent** or **transient**.

Before proving this theorem, let us introduce the following definitions. First, fix a state \(i \in I\). Let \(R_i^{(0)} \equiv 0\) be the first hitting time to state \(i\). For \(m \geq 1\),
\[
R_i^{(m)} := \inf\{t > R_i^{(m-1)} \mid X_t = i\}.
\]

denote the \(m\)-th **return time** to state \(i\) (previously \(T_m\) for \(A = \{i\}\)). We use capital \(R\) to emphasize the “return time” to a state in the Markov chain. This definition is illustrated in Figure 9.1. If the chain starts at state \(i\), we count the \(t = 0\) step as a “visit” or ”hit”, but we do not count it as a “return”!

The following theorem states that the length between two return times is independent of the past information.

**Theorem 9.3.** Given \(R_i^{(m)} < \infty\), the r.v. \(G_i^{(m)} = R_i^{(m+1)} - R_i^{(m)} \) (\(m\)-th gap) is independent of \(\mathcal{F}_{R_i^{(m)}}\) for all \(m \geq 0\) and \(i \in I\).
Proof. Given \( R_i^{(m)} < \infty \), we use the strong Markov Property to write
\[
\left( X_{R_i^{(m)} + k} \right)_{k \geq 0} \text{ is independent of } \mathcal{F}_{R_i^{(m)}}.
\]

Using the definition, we rewrite the gap as
\[
G_i^{(m)} = R_i^{(m+1)} - R_i^{(m)} = \inf \left\{ t > R_i^{(m)} \mid X_t = i \right\} - R_i^{(m)} = \inf \left\{ t = R_i^{(m)} + k \mid k > 0, \ X_t = i \right\} - R_i^{(m)} = \inf \left\{ t > 0 \mid X_{R_i^{(m)} + t} = i \right\},
\]
where \( \inf \left\{ t > 0 \mid X_{R_i^{(m)} + t} = i \right\} \) is independent of \( \mathcal{F}_{R_i^{(m)}} \) given \( R_i^{(m)} < \infty \) and \( X_i^{(m)} = i \). \[\blacksquare\]

**Corollary 9.4.** The r.v.s \((G_i^{(m)}, m \geq 0)\) are i.i.d. and we have \( G_i^{(0)} \) has the same distribution as \( R_i^{(1)} \) as long as \( G_i^{(m)} < \infty \). Moreover, \( R_i^{(m)} \) is sum of \( m \) many i.i.d. random variables.

The indicator function \( \mathbb{1}_{X=j} \) is the random variable equal to 1 if \( X = j \) and 0 otherwise. Let us introduce the number of visits written in terms of indicator functions.

**Definition 9.5.** The total number of returns \( V_i \) to \( i \) is defined as
\[
V_i = \sum_{t=0}^{\infty} \mathbb{1}_{(X_t = i)}.
\]

As a result of definition 9.5, it follows from Fubini’s theorem that
\[
E_i(V_i) = \sum_{n=0}^{\infty} E_i(\mathbb{1}_{(X_n = i)}) = \sum_{n=0}^{\infty} P_i(X_n = i) = \sum_{n=0}^{\infty} p^{(n)}_{ii}.
\]

We can also show that the following relation is true. For \( m = 0, 1, \ldots \) we have
\[
P_i(V_i > m) = P_i\left( G_i^{(j)} < \infty, \ j = 0, 1, 2, \cdots, m - 1 \right) \quad \text{(from corollary 9.4)}
\]
\[
= \prod_{j=0}^{m-1} P_i\left( G_i^{(j)} < \infty \right) = P_i\left( R_i^{(1)} < \infty \right)^m
\]
Definition 9.6. We define, for \( i \in I \),
\[
a_i = \mathbb{P}_i \left( R_i^{(1)} < \infty \right).
\]

From definition 9.6 we can write the expectation of the number of returns in terms \( a_i \) as
\[
E(V_i) = \sum_{m=0}^{\infty} \mathbb{P}_i (V_i > m) = \sum_{m=0}^{\infty} a_i^m = \begin{cases} 
\infty & \text{if } a_i = 1 \\
\frac{1}{1 - a_i} & \text{if } a_i < 1.
\end{cases}
\]

Moreover, we have the following relation:
\[
\mathbb{P}(V_i = \infty) = \lim_{m \to \infty} \mathbb{P}(V_i > m) = \begin{cases} 
1 & \text{if } a_i = 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Using the relations derived above, we are ready to prove Theorem 9.2

**Proof.** We prove that the following dichotomy holds:

**Case I:** When \( \mathbb{P} \left( R_i^{(1)} < \infty \right) = 1 \), then \( E_i(V_i) = \sum_{n \geq 0} p^{(n)}_{ii} = \infty \) and \( \mathbb{P}_i (V_i = \infty) = \lim_{m \to \infty} \mathbb{P}_i (V_i > m) = 1 \), which proves that \( i \) is recurrent.

**Case II:** When \( \mathbb{P} \left( R_i^{(1)} < \infty \right) < 1 \), we have \( E_i(V_i) = \sum_{n \geq 0} p^{(n)}_{ii} = \frac{1}{1 - \mathbb{P}_i \left( R_i^{(1)} < \infty \right)} < \infty \), and \( \mathbb{P}_i (V_i = \infty) = 0 \), which proves that \( i \) is transient.

With these results, we are ready to completely solve the problem of recurrence or transience for Markov chains with finite state space. We now show that recurrence and transience are class properties.

**Lemma 9.7.** For any communication class, all states are recurrent, or all states are transient.

**Proof.** Let \( C \) be a communication class. Take \( i, j \in C \). It is enough to prove that if \( i \) is recurrent, so is \( j \). Now,
\[
i \text{ is recurrent } \iff \sum_{n=0}^{\infty} p^{(n)}_{ii} = \infty.
\]

Since \( i \leftrightarrow j \), then there exists \( s \) and \( t \) such that \( p^{(s)}_{ij} > 0 \) and \( p^{(t)}_{ji} > 0 \). We have
\[
\sum_{n=0}^{\infty} p^{(n)}_{jj} \geq \sum_{n=0}^{\infty} p^{(t+n+s)}_{jj} \geq \sum_{n=0}^{\infty} p^{(t)}_{ji} p^{(s)}_{ii} p^{(s)}_{ij} = p^{(t)}_{ji} p^{(s)}_{ij} \sum_{n=0}^{\infty} p^{(n)}_{ii} = \infty.
\]

Thus, we have that \( j \) is recurrent.

**Lemma 9.8.** Any recurrent communication class is closed.

**Proof.** Let \( C \) be a recurrent communication class. Suppose \( C \) is not closed. Then there exists \( i, j \) such that \( j \notin C \), \( i \in C \) and \( i \to j \), i.e., there exists \( m \) such that \( p^{(m)}_{ij} > 0 \) and
1. \( P_i(X_n = i, \text{ infinitely often}) = 1 \) by assumption,
2. \( P_j(X_n = i) = 0 \) for all \( n \).

We can conclude that \( P_i(\{X_m = j\} \cap \{X_n = i \text{ infinitely often}\}) > 0 \). Remember that for the events \( A \) and \( B \) where \( P(A) = 1 \), we have \( P(A \cup B) = 1 \) and thus \( P(A \cap B) = P(B) \). Thus,

\[
0 < P_i(\{X_m = j\} \cap \{X_n = i \text{ infinitely often}\}) \\
= P_i(\{X_m = j\} \cap \{X_{m+n} = i \text{ infinitely often}, n \geq 0\}) \\
= P_i(X_m = j) P_j(X_n = i \text{ infinitely often}), \text{ by the Markov Property} \\
= 0 \quad \text{which follows from item 2}
\]

and we arrive at a contradiction. \( \blacksquare \)

Lemma 9.9. Any finite irreducible Markov chain is recurrent.

Proof. We prove this lemma using the pigeonhole principle. First fix a state \( i \). Since the state space is finite, we have

\[ P_i(V_j = \infty) > 0 \text{ for some } j. \]

Also, \( j \to i \Rightarrow \exists m \text{ such that } p_{ij}^{(m)} > 0 \). Hence,

\[ P_i(V_j = \infty) > 0 \Leftrightarrow j \text{ is recurrent } \Rightarrow \text{ entire class is recurrent.} \]

This completes the proof. \( \blacksquare \)

Example 9.10. What happens if the Markov chain is infinite? Figure 9.2 is an irreducible, not recurrent, Markov chain. We have that

\[ P_k(T_0 < \infty) = (q/p)^k < 1 \]

\[
\begin{array}{c}
\text{Figure 9.2: } q < p \Rightarrow \text{is transient (left as exercise)}
\end{array}
\]

Lemma 9.11. If a communicating class is recurrent, then \( P_i(T_j < \infty) = 1 \) for all \( i, j \).