Homework 7

MATH 564/STAT 555/MATH 466: Applied Stochastic Processes

Due date: December 06, 2022

For Math 466, skip questions marked with *; the total point is 25. For Math 564/Stat 555, answer all questions; the total point is 30. Write your name and course number (i.e., Math 564, Math 466, Stat 555) clearly on top of the first page. You can work in groups or on your own. Please indicate whom you worked with; it will not affect your grade in any way.

1. (3+2 pts) Customers arrive at a single-server queue in a Poisson stream of rate $\lambda$. Each customer has a service requirement distributed as the sum of two independent exponential random variables of parameter $\mu$. Service requirements are independent of one another and of the arrival process.

(a) Write down the generator matrix $Q$ of a continuous-time Markov chain which models this, explaining what the states of the chain represent.

(b) Argue that the chain is transient if $\lambda > \mu/2$.

**Solution:**

(a) Let $X_t$ be the number of customers in the queue at time $t \geq 0$ and $Y_t$ be the indicator random variable that a customer is in the first part of the service at time $t$.

We define $Z_t = 2X_t + Y_t$, for $t \geq 0$, then $(Z_t)_{t \geq 0}$ becomes a Markov process with state space $I = \{1, 2, 3, \ldots\}$ and rate matrix $Q = ((q_{ij}))_{i,j \in I}$ where

$$
q_{1,1} = -\lambda,
q_{i,i} = -\lambda - \mu \quad \text{for } i = 2, 3, \ldots,
q_{i,i+2} = \lambda \quad \text{for } i = 1, 2, 3, \ldots,
q_{i,i-1} = \mu \quad \text{for } i = 2, 3, \ldots.
$$

We can get the number of customers at time $t$ as $X_t = \text{integer part of } Z_t/2$.

(b) Define $p = \lambda/(\lambda + \mu), q = 1 - p = \mu/(\lambda + \mu)$. Then the jump chain $U_n, n \geq 0$ is a random walk with step size distribution $+2$ w.p. $p$ and $-1$ w.p. $q$, except $1 \rightarrow 3$ w.p. 1. Note that, $U_n \rightarrow \infty$ as $n \rightarrow \infty$ if expected step size $= 2p - q > 0$ or $\lambda > \mu/2$. Thus the jump chain is transient and hence the continuous time Markov chain is also transient.

How to generalize the problem if the service time is sum of $k$ many independent Exponential random variables?

2. (3+2+2+3* pts) Fix an integer $k \geq 2$. Suppose that, each bacterium in a colony splits into $k$ many identical bacteria after an exponential time of parameter $\lambda$, which then split in the same way but independently. Let $X_t$ denote the size of the colony at time $t$, and suppose $X_0 = 1$.

(a) Show that the probability generating function $\varphi(t) = E(z^{X_t})$ satisfies, for fixed $z \in (0, 1)$,

$$
\varphi(t) = ze^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} \varphi(s)^k ds.
$$

(b) Deduce from a) that $\frac{d}{dt}\varphi(t) = -\lambda \varphi(t)^k (\varphi(t)^{1-k} - 1)$. Solve to get a formula for $\varphi(t)$.
c) For $k = 2$ show that, we have
\[ P(X_t = n) = (1 - q)q^{n-1}, \quad n = 1, 2, \ldots \]
where $q = 1 - e^{-\lambda t}$.

d) Using b) with general $k \geq 2$, show that the random variable
\[ \eta_t := \frac{1}{k - 1} e^{-(k-1)\lambda t} \cdot (X_t - 1) \]
converges in distribution to Gamma distribution with shape parameter $1/(k-1)$ and scale parameter 1, as $t \uparrow \infty$.

**Solution:** a) Let $T$ be the time of the first split. Thus $T$ has an exponential distribution with rate $\lambda$. Then,
\[
\varphi(t) = \mathbb{E}(z^{X_t} 1_{T>t}) + \mathbb{E}(z^{X_t} 1_{T\leq t}) = ze^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \mathbb{E}(z^{X_t} | T = s) \, ds
\]
\[
= ze^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \varphi(t-s)^k \, ds.
\]
In the last line, we used that $\mathbb{E}(z^{X_t} | T = s) = \mathbb{E}(z^{X_1} + X_2 + \cdots + X_{k-1}) = \mathbb{E}(z^{X_{t-s}})^{k} = \varphi(t-s)^2$, where $X_1, X_2, \ldots, X_k$ is an independent copy of $X$.

b) A change of variable and multiplication by $e^{\lambda t}$ gives
\[
e^{\lambda t} \varphi(t) = z + \int_0^t \lambda e^{\lambda u} \varphi(u)^k \, du.
\]
Taking derivative in $t$ yields
\[
e^{\lambda t} \varphi'(t) + \lambda e^{\lambda t} \varphi(t) = \lambda e^{\lambda t} \varphi(t)^k,
\]
which gives the required
\[
\varphi'(t) = -\lambda \varphi(t)^k (\varphi(t)^{1-k} - 1) \text{ or } (\varphi(t)^{1-k})' = (k-1)\lambda (\varphi(t)^{1-k} - 1).
\]
By integrating this differential equation and using the initial condition $\varphi(0) = z$, we get that
\[
\log(\varphi(t)^{1-k} - 1) = (k-1)\lambda t + \log(z^{1-k} - 1)
\]
or
\[
\varphi(t) = \frac{ze^{-\lambda t}}{(1 - z^{k-1})^{1/(k-1)}(1 - e^{-(k-1)\lambda t})^{1/(k-1)}}.
\]
c) For $k = 2$, we get with $q = 1 - e^{-\lambda t}$
\[
\varphi(t) = \frac{z(1 - q)}{1 - zq} = \sum_{n=1}^{\infty} (1 - q)q^{n-1}z^n.
\]
Since $\varphi(t) = \mathbb{E}(z^{X_t}) = \sum_{n=1}^{\infty} z^n P(X_t = n)$, the claim follows.
3. (3+2 pts) Let \((X_n)_{n \geq 0}\) be a Markov chain on the state space \(I\) and let \(A\) be an absorbing set in \(I\). Define

\[ T := \inf\{n \geq 0 \mid X_n \in A\} \]

as the hitting time for \(A\) and

\[ h(i) := \mathbb{P}_i(T < \infty). \]

(a) Show that the process \(M_n := h(X_n), n \geq 1\) is a martingale w.r.t the standard filtration.

(b) Note that, \(h(X_T) = 1\) as \(h(i) = 1\) for \(i \in A\), but \(h(i)\) need not be 1 for \(X_0 = i \notin A\). Why can’t we use Optional stopping theorem here? Explain.

**Solution:** (a) Clearly \(M_n\) is integrable (bounded by 1), adapted to the standard filtration \(\mathcal{F}_n = \sigma(X_0, X_1, X_2, \ldots, X_n)\) for \(n \geq 0\). Moreover, \(h(i) = 1\) for \(i \in A\) and for \(i \notin A\) we have

\[ h(i) = \mathbb{P}_i(T < \infty) = \sum_{j \in I} p_{ij} \mathbb{P}_j(T < \infty) = \sum_{j \in I} p_{ij} h(j). \]

Also, \(A\) is absorbing and thus \(X_n \in A\) implies \(X_{n+1} \in A\). Thus,

\[ \mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = \mathbb{E}(h(X_{n+1}) \mid X_n) = 1_{X_n \in A} \cdot 1 + 1_{X_n \notin A} \cdot \sum_{j \in I} p_{X_n,j} h(j) \]

\[ = 1_{X_n \in A} \cdot h(X_n) + 1_{X_n \notin A} \cdot h(X_n) = h(X_n) = M_n \]

for all \(n \geq 0\).

(b) Take \(X_0 = i\). If \(i \in A\), \(T = 0\) and \(M_T = M_0\). Now assume that \(i \notin A\). If \(h(i) = 1\), i.e., \(\mathbb{P}_i(T < \infty) = 1\), then we can use OST condition ii) to get \(\mathbb{E}(M_T) = 1 = \mathbb{E}(M_0)\). However, if \(h(i) < 1\) then \(\mathbb{E}_i(T < \infty) < 1\) and we cannot use OST part (ii) any more; indeed we have \(T = \infty\) with positive probability and hence \(M_T\) is not even defined in that case.

4. (3+3+2+2* pts) Calls arrive at a telephone exchange as a Poisson process of rate \(\lambda\), and the lengths of calls are independent exponential random variables of parameter \(\mu\).

(a) Assuming that infinitely many telephone lines are available, set up a Markov chain model for this process. What is this model called in literature.

(b) Show that for large \(t\) the distribution of the number of lines in use at time \(t\) is approximately Poisson with mean \(\lambda/\mu\).

(c) Find the mean length of the busy periods during which at least one line is in use.

(d) Show that, \(m(t)\) = the expected number of lines in use at time \(t\), given that \(n\) are in use at time 0, is

\[ m(t) = ne^{-\mu t} + \frac{\lambda}{\mu} (1 - e^{-\mu t}). \]

**Solution:** (a) Consider an \(M/M/\infty\) queue \((X_t)_{t \geq 0}\) with Poisson rate \(\lambda\) arrival, Poisson rate \(\mu\) service and \(\infty\) many servers. Compute the \(Q\) matrix.
(b) Show that (using detailed balance) that the invariant distribution os Poisson($\lambda/\mu$). Thus, by convergence to equilibrium theorem we have the result.

(c) The expected return time starting from an empty queue is $m_0$. Out of this the expected time spent at state 0 is $1/q_0$. Thus, the answer is $m_0 - 1/q_0 = (1/\pi_0 - 1)/q_0 = (e^{\lambda/\mu} - 1)/\lambda$.

(d) Out of $n$ initial customers, number of customers still in service at time $t$ is $ne^{-\mu t}$ on average. Let $m_0(t) = E_0(X_t)$. Then it is easy to see that $m(t) = ne^{-\mu t} + m_0(t)$. Moreover, we have conditional on the first customer arrival time
\[
m_0(t) = \int_0^t \lambda e^{-\lambda s} (e^{-\mu (t-s)} + m_0(t-s)) \, ds = \lambda e^{-\lambda t} \int_0^t (e^{(\lambda-\mu)s} + e^{\lambda s} m_0(s)) \, ds.
\]
Solving we get the result.