Homework 4

MATH 564/STAT 555/MATH 466: Applied Stochastic Processes

Due date: October 20, 2022

For Math 466, skip questions marked with ∗; the total point is 25. For Math 564/Stat 555, answer all questions; the total point is 30. Write your name and course number (i.e., Math 564, Math 466, Stat 555) clearly on top of the first page. You can work in groups or on your own. Please indicate whom you worked with; it will not affect your grade in any way.

1. (5 × 2 pts.) In each of the following cases determine whether the stochastic matrix $P$ on the state space $I$, which you may assume irreducible, is reversible:
   (a) \[
   \begin{bmatrix}
   1 - a & a \\
   b & 1 - b
   \end{bmatrix}
   \]
   (b) \[
   \begin{bmatrix}
   0 & p & 1 - p \\
   1 - p & 0 & p \\
   p & 1 - p & 0
   \end{bmatrix}
   \]
   (c) $I = \{0, 1, \ldots, N\}$ and $p_{ij} = 0$ if $|j - i| \geq 2$ and $p_{ij} > 0$ if $|j - i| \leq 1$.
   (d) $I = \{0, 1, \ldots\}$ and $p_{01} = 1, p_{i,i+1} = p, p_{i,i-1} = 1 - p$ for $i \geq 1$.
   (e) $p_{ij} = p_{ji}$ for all $i, j \in I$.

2. (3 + 3∗ pts.) Let $(X_n)_{n \geq 0}$ be an irreducible Markov chain on the state space $I$ having transition matrix $P$ and an invariant distribution $\pi$. For $A \subseteq I$ let $(Y_m)_{m \geq 0}$ be the random process on $A$ obtained by observing $(X_n)_{n \geq 0}$ whilst in $A$. More precisely, if $T_0 := \min\{n \geq 0 \mid X_n \in A\}$, and $T_m := \min\{n > T_{m-1} \mid X_n \in A\}$, for $m \geq 1$,
   let $Y_m = X_{T_m}$.
   (a) Prove that $(Y_m)_{m \geq 0}$ is a Markov chain and compute its transition probabilities in terms of $P$.
   (b) Show that $(Y_m)_{m \geq 0}$ is positive recurrent and find its invariant distribution.

3. (3 + 2∗ pts.) Each morning a student takes one of the $k$ books he owns from his shelf. The probability that he chooses book $i$ is $\alpha_i$, where $0 < \alpha_i < 1$ for $i = 1, \ldots, k$, and choices on successive days are independent. In the evening he replaces the book at the left-hand end of the shelf. Let $p_n$ denote the probability that on day $n$ the student finds the books in the correct order 1, 2, . . . from left to right.
   (a) Show that, irrespective of the initial arrangement of the books, $p_n$ converges as $n \to \infty$.
   (b) Determine the limit when $k = 3$.

4. (2+1 pts.) Happy Harry used to play semipro basketball where he was a defensive specialist. His scoring productivity per game fluctuated between three states: 1 (scored 0 or 1 points), 2 (scored between 2 and 5 points), 3 (scored more than 5 points). Inevitably, if Harry scored a lot of points in one game, his jealous teammates refused to pass the ball in the next game, so his productivity it
next game was nil. The team statistician, Mrs. Doc, upon observing the transitions between states, concluded these transitions could be modeled by a Markov chain with transition matrix

\[
P = \begin{bmatrix}
0 & 1 & \frac{2}{3} \\
\frac{1}{7} & 0 & \frac{2}{7} \\
1 & 0 & 0
\end{bmatrix}
\]

(a) What is the long run proportion of games that Harry had high scoring game?

(b) The salary structure in the semipro leagues includes incentives for scoring. Harry was paid $500/game for a high scoring performance, $300/game when he scored between 2 and 5 points, and only $100/game when he score nil. What was Harry’s long run earning rate?

5. (3+2+1 pts.) Consider the two state Markov chain with state \( \{1, 2\} \) and with transition probabilities \( p_{12} = p \in [0, 1], p_{21} = q \in (0, 1) \). Let us assume that \( 0 < q < 1 \) is fixed but we can choose \( p \in [0, 1] \) however we like. Moreover, assume that there is a payoff of \( r > 0 \) every time we visit state 2, and a cost of \( c(p) > 0 \) every time we visit state 1. Then:

(a) Compute the long-term profit per time step as a function of \( p \). Describe a method to find the optimal choice of \( p \) to maximize profits. Does such an optimal choice always exist?

(b) Assume that the cost function is linear, \( i.e., c(p) = \alpha p \) for some \( \alpha > 0 \). Show that the optimal choice of \( p \) will lead to a profit-per-step of

\[
\left( \frac{r - \alpha q}{1 + q} \right)^+ 
\]

where we denote \( x^+ := \max\{x, 0\} \).

(c) Assume that the cost function is constant, \( i.e., c(p) = c \) for all \( p \). What is the optimal profit?