Homework 3
MATH 564/STAT 555/MATH 466: Applied Stochastic Processes
Due date: October 6, 2022

For Math 466, skip questions marked with ∗; the total point is 23. For Math 564/Stat 555, answer all questions; the total point is 30. Write your name and course number (i.e., Math 564, Math 466, Stat 555) clearly on top of the first page. You can work in groups or on your own. Please indicate whom you worked with; it will not affect your grade in any way.

1. (2+1+1 pts.) Let $P$ be an $N \times N$ transition matrix which is irreducible. Show that, for any $\alpha \in (0, 1)$, the $\alpha$–lazy version of the chain, defined as
   $$\tilde{P} := \alpha P + (1-\alpha)I_N,$$
   where $I_N$ is the $N \times N$ identity matrix, is irreducible and aperiodic. Moreover, show that if $\pi$ is an invariant distribution for $P$, then it is also invariant for the $\alpha$–lazy version of $P$.

   **Solution:** Clearly, all entries in $\tilde{P}$ are non-negative and the row sums are one. Thus $\tilde{P}$ is a stochastic matrix. Moreover, $\tilde{p}_{ij} \geq \alpha p_{ij} + (1-\alpha)\delta_{ij}$ for all $i, j$. Thus $\tilde{p}^n_{ij} \geq \alpha^n p^n_{ij}$ for all $n \geq 1, i, j \in I$ and $i \rightarrow j$ for $P$ implies $i \rightarrow j$ for $\tilde{P}$. Since, $P$ is irreducible, this implies that $\tilde{P}$ is irreducible. Now $\tilde{p}_{ii} \geq 1-\alpha > 0$ for all $i$, thus $\tilde{P}$ is aperiodic. Moreover, $\pi^\top P = \pi^\top$ implies that
   $$\pi^\top \tilde{P} = \alpha \pi^\top P + (1-\alpha)\pi^\top I = \alpha \pi^\top + (1-\alpha)\pi^\top = \pi^\top.$$
   This completes the proof.

2. (3 pts) Let $P$ be an $N \times N$ transition matrix which is irreducible and aperiodic. Perron-Frobenius theorem states that 1 is the unique largest eigenvalue of $P$ and for all other eigenvalues $\lambda$ of $P$ we have $|\lambda| < 1$. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_N)^\top$ be the left eigenvector of $P$ corresponding to 1 such that $\sum_{i=1}^N \pi_i = 1$. Use the above result and linear algebra techniques (e.g. Jordan Canonical Form) to prove that as $t \rightarrow \infty$,
   $$(P^t)_{ij} \rightarrow \pi_j$$
   for all $i, j$.

   **Solution:** If the matrix $P$ is diagonalizable, then we can find an $N \times N$ matrix $S$ whose rows form a right eigenbasis and we can write
   $$P = S^{-1}DS$$
   where $D$ is a diagonal matrix consisting of right eigenvalues of $P$. However, in general $P$ is not diagonalizable and we will use Jordan Canonical form for the proof.
   Let $\lambda_1 = 1, \lambda_2, \lambda_3, \ldots, \lambda_k$ be all the distinct (right) eigenvalues of $P$. Let $d_i$ be the geometric multiplicity of $\lambda_i$ for $i \geq 1$. By Perron-Frobenius theorem we have $|\lambda_i| < 1$ for all $i > 1$ and $d_1 = 1$. By Jordan decomposition we can write $P = S^{-1}JS$ where $J$ is a block diagonal matrix
   $$\text{diagonal}(J_{1,1}, J_{2,2}, \ldots, J_{2,d_2}, \ldots, J_{k,k}, \ldots, J_{k,d_k})$$
   where
with each $J_{i,j}$ has $\lambda_j$ in the diagonal, 1 in the upper diagonal line and zero everywhere else. In particular $J_{1,1} = |1|$ and thus $SP = JS$ implies that the first row of $S$ is a non-zero multiple of $\pi^T$, $PS^{-1} = S^{-1}J$ implies that the first column of $S^{-1}$ is a non-zero multiple of $1$.

Now, we have $P^n = S^{-1}J^nS$ for all $n \geq 1$. Let $\theta := \max_{i \geq 2} |\lambda_i| < 1$. It is easy to check that $(J^n)_{11} = 1, (J^n)_{ij} = 0$ for $i > j$ and $(J^n)_{ij} \leq \theta^{n-1}$ for all other $i \leq j$. Thus, $J^n \to E_{11}$ as $n \to \infty$, where $E_{11}$ is the matrix with 1 at the $(1,1)$-th entry and zero everywhere else. In particular, we have

$$\lim_{n \to \infty} P^n = S^{-1}E_{11}S = I\pi^T.$$

3. (1 + 1 + 2 + 2 + 2 + 2 + 2 + 2 + 2 pts) A particle moves in the $d$-dimensional hypercube $\{+1, -1\}^d$ in the following way: at $n$-th step the particle is equally likely to flip the sign of one of the co-ordinates in its position vector $X_n \in \{+1, -1\}^d$, independent of its past motion. Let $i$ be the initial vertex occupied by the particle, $o = -1$ be the vertex opposite to $i$. Answer the following questions:

(a) Show that $(X_n)_{n \geq 0}$ is an irreducible Markov chain.

(b) When is the chain aperiodic.

(c) What are the invariant distributions?

(d) Find the expected number of steps until the particle returns to $i$.

(e) *Find the expected number of visits to $o$ until the first return to $i$.

(f) Let $Y_n = \text{sum of the co-ordinates in } X_n$. Show that $(Y_n)_{n \geq 0}$ is a Markov Chain. Find its state space and transition matrix.

**Solution:** (a) We want to show that any vertex leads to any other vertex and it is enough to show that any vertex leads to the vertex $1 = (+1, +1, \ldots, +1)$. Take a vertex $x = (x_1, x_2, \ldots, x_d) \in \{+1, -1\}^d$ such that $x \neq 1$. Let $i_1 < i_2 < \cdots < i_k, k \in \{1, 2, \ldots, d\}$ be such that $\{i \mid x_i = -1\} = \{i_1, i_2, \ldots, i_k\}$ (the co-ordinates where the value is $-1$). Define the $k + 1$ many intermediate vertices, for $j = 0, 1, \ldots, k$ by

$$x^{(j)}_i = \begin{cases} -1 & \text{if } i \in \{i_{j+1}, i_{j+2}, \ldots, i_k\} \\ +1 & \text{otherwise.} \end{cases}$$

Clearly, $x^{(0)} = x, x^{(k)} = 1$ and $p_{x^{(j)}, x^{(j+1)}} = \frac{1}{d}$ for all $j = 0, 1, \ldots, k$. Hence

$$p_{x, 1}^{(k)} \geq \prod_{j=0}^{k-1} p_{x^{(j)}, x^{(j+1)}} > 0$$

(b) By parity consideration, the chain is periodic with period 2. Number of +1 co-ordinates always changes from odd to even or even to odd in one step of the Markov chain.

(c) The chain is a finite irreducible Markov chain and hence is positive recurrent. Also, the transition matrix is doubly stochastic and hence the unique invariant distribution is the uniform distribution on the hypercube with probability $\pi_x = 2^{-d}$ on each vertex $x$.

(d) We have $\pi_1 = 1/m_1$. Thus the expected number of steps until the particle returns to $i$ starting from $1$ is $m_0 = 2^d$.

(e) This is same as calculating $\gamma^{(i)}_o$. But, $\gamma^{(i)}_o = \pi_o/\pi_1 = 1$.

(f) The possible values for $Y_n, i.e.,$ the state space is $I = \{d - 2i \mid i = 0, 1, \ldots, d\}$. The proof is
similar to 3.b) and hence skipped. The transition matrix for $Y_n$ is $P' = (p'_{a,b})_{a,b \in I}$ where

$$p'_{a,b} = \begin{cases} \frac{d-a}{d} & \text{if } b - a = +2 \\ \frac{d+a}{d} & \text{if } b - a = -2 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } a, b \in I.$$ 

4. $(2 + 2 + 3^* + 1 \text{ pts})$ Consider the “random walk on the $d$-regular infinite tree”. Define the vertices of the tree $T$ by all vectors of any length whose elements are in the set $\{1, 2, \ldots, d\}$, i.e.,

$$T = \bigcup_{n=0}^{\infty} \{1, 2, \ldots, d\}^n.$$ 

The unique zero length element is called the root and is denoted by $\emptyset$. Choose $p_k > 0$ for $k = 0, 1, 2, \ldots, d$ such that $\sum_{i=0}^{d} p_i = 1$. We then define the transition probabilities as

$$P_{k_1 k_2 \cdots k_n, k_1 k_2 \cdots k_n, \ell} = p_{\ell} \quad \text{for } \ell = 1, 2, \ldots, d$$

$$P_{k_1 k_2 \cdots k_n, k_1 k_2 \cdots k_{n-1}} = p_0$$

for any non-root vertices and define

$$P_{\emptyset, \ell} = p_{\ell} \quad \text{for } \ell = 1, 2, \ldots, d$$

$$P_{\emptyset, \emptyset} = p_0.$$ 

So from any node the Markov chain goes to its $i$-th children with probability $p_i$ and to its parent node (or itself if no parent) with probability $p_0$. See Figure 1 for a pictorial representation for $d = 2$. Answer the following questions.

(a) Prove that the above Markov Chain $(X_t)_{t \geq 0}$ on the state space $T$ is irreducible and aperiodic.

(b) We define the length of a vertex in $T$ to be $n$ if it is in $\{1, 2, \ldots, d\}^n$. Show that $(Y_t = \text{length of } X_t)_{t \geq 0}$ is a Markov Chain. Find its transition matrix.

(c) *What are the conditions on $p_k$ such that the chain $(X_t)_{t \geq 0}$ is recurrent. You can use the results from the Gambler’s ruin problem.

(d) Assume that we have to take $p_0 = p_1 = \cdots = p_d = \frac{1}{1+d}$. For what values of $d$ is this recurrent.
5. (3 pts) A fair die is thrown repeatedly. Let $X_n$ be the sum of the first $n$ throws. Find \[ \lim_{n \to \infty} \mathbb{P}(X_n \text{ is a multiple of } 9). \]

**Solution:** Let $Y_n = X_n \mod 9, n \geq 0$. Then $Y = (Y_n)_{n \geq 0}$ is a Markov chain (as $(a+b) \mod 9 = (a \mod 9 + b) \mod 9$ ) with state space $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. For each state $i \in S$ only eight successive states can be reached in one step. Hence, each row of the transition matrix $P$ has eight entries equal to 1/8 and one diagonal entry equal to 0. Same argument shows that each column of $P$ has eight entries equal to 1/8 and one entry equal to 0. Therefore, $P$ is doubly stochastic. It follows that the unique stationary distribution of $Y$ is $\pi = (1/9, 1/9, \ldots, 1/9)$. Clearly $Y$ is irreducible ($p_{ij} = \frac{1}{6}$ for $i \neq j$) and aperiodic (e.g., $p_{00}^2 \geq p_{01}p_{10} > 0$ and $p_{00}^3 \geq p_{01}p_{12}p_{20} > 0$). Thus we have

\[ \lim_{n \to \infty} \mathbb{P}(X_n \text{ is a multiple of } 9) = \lim_{n \to \infty} \mathbb{P}(Y_n = 0) = \frac{1}{9}. \]
6. (2* pts) Let \((X_t)_{t \geq 0} \sim \text{Markov}(\lambda, P)\) on the state space \(I\) and \((Y_t)_{t \geq 0} \sim \text{Markov}(\mu, P)\) on the same state space. Assume that \(W_t = (X_t, Y_t), t \geq 0\) is a recurrent Markov chain. Show that for any \(j \in I\) we have
\[
\lim_{n \to \infty} |P(X_n = j) - P(Y_n = j)| = 0.
\]

**Solution:** The proof is a modification of the proof of “Convergence to equilibrium theorem” (Lecture 12). The second step in the proof was to show that
\[
T := \inf\{n \geq 0 \mid X_n = Y_n = b\} = \inf\{n \geq 0 \mid W_n = (b, b)\}
\]
is finite a.s. for a fixed \(b \in I\). Here this is immediate since the product chain is assumed to be recurrent. The rest of the proof is the same.