Homework 2
MATH 564/STAT 555/MATH 466: Applied Stochastic Processes
Due date: September 22, 2022

For Math 466, skip questions marked with ∗; the total point is 20. For Math 564/Stat 555, answer all questions; the total point is 30. Write your name and course number (i.e., Math 564, Math 466, Stat 555) clearly on top of the first page. You can work in groups or on your own. Please indicate whom you worked with; it will not affect your grade in any way.

1. (3 pts.) Suppose that $(X_n)_{n \geq 0}$ is Markov($\lambda, P$). Define $Y_n = X_{kn}$ for some $k \geq 1$. Show that $(Y_n)_{n \geq 0}$ is also a Markov chain. Find the initial distribution and transition matrix.

Solution: The assumption that $(X_n)_{n \geq 0}$ is Markov($\lambda, P$) implies that

$$P(X_i = x_i, i = 0, 1, \ldots, n) = \lambda x_0 \prod_{i=1}^{n} p_{x_{i-1}x_i}$$

for all $n \geq 1, x_0, x_1, \ldots, x_n \in I$ where $I$ is the state space and $P = ((p_{ij}))_{i,j \in I}$ is the transition matrix. In particular, using the fact that

$$p^{(k)}_{ij} := (P^k)_{ij} = \sum_{x_1, x_2, \ldots, x_{k-1} \in I} p_{ix_1}p_{x_1x_2}\cdots p_{x_{k-1}j}$$

and summing over, we have

$$P(Y_i = y_i, i = 1, 2, \ldots, n) = P(X_{ki} = y_i, i = 0, 1, \ldots, n) = \lambda y_0 \prod_{i=1}^{n} p^{(k)}_{y_{i-1}y_i}$$

for all $n \geq 1, y_0, y_1, \ldots, y_n \in I$. Thus we have, $(Y_n)_{n \geq 0}$ is Markov($\lambda, P^k$).

2. (3 pts.) Show that for any $i, j \in I, A \subseteq I$ and $i \notin A$ where $I$ is the state space, we have

$$P_i(H^A < \infty | X_1 = j) = P_j(H^A < \infty)$$

and

$$E_i(H^A | X_1 = j) = 1 + E_j(H^A).$$

Solution: Fix $A \subseteq I, i, j \in I, i \notin A$. We have

$$H^A := \inf\{n \geq 0 \mid X_n \in A\}.$$

Thus, when $X_0 = i \notin A$, we have

$$H^A = \inf\{n \geq 1 \mid X_n \in A\} = 1 + \inf\{n \geq 0 \mid X_{n+1} \in A\}.$$

Hence,

$$P_i(H^A < \infty | X_1 = j) = P(\inf\{n \geq 0 \mid X_{n+1} \in A\} < \infty | X_1 = j, X_0 = i) = P(\inf\{n \geq 0 \mid X_n \in A\} < \infty | X_0 = j) = P_j(H^A < \infty)$$
where in the second equality we used Markov property and the fact that \( \inf\{n \geq 0 \mid X_{n+1} \in A\} \) depends only on \( X_k, k \geq 1 \). Taking expectations, we also get

\[
E_i(H^A \mid X_1 = j) = E(1 + \inf\{n \geq 0 \mid X_{n+1} \in A\} \mid X_1 = j, X_0 = i)
\]

\[
= 1 + E(\inf\{n \geq 0 \mid X_n \in A\} \mid X_0 = j) = 1 + E_j(H^A).
\]

3. (1+2+1+3+2* pts.) We consider a Markov chain which is given by a unidirectional ring with one escape point, i.e., pick an integer \( N > 1 \) and \( 0 \leq p \leq 1 \), and consider the Markov chain with state space \( I = \{1, 2, \ldots, N + 1\} \) and transition probabilities

\[
p_{1,1} = 0, \quad p_{1,2} = p, \quad p_{1, N+1} = 1 - p,
\]

\[
p_{i,i+1} = p, \quad p_{i,i} = 1 - p, \quad \text{for } i = 2, 3, \ldots, N - 1
\]

\[
p_{N,1} = p, \quad p_{N,N} = 1 - p,
\]

\[
p_{N+1,N+1} = 1.
\]

For \( N = 4 \), the corresponding graph is:

\[
\begin{array}{c}
1 \quad 5 \quad 1 - p \\
2 \\
3 \quad 1 - p \\
4 \quad 1 - p
\end{array}
\]

Denote \( A = \{N + 1\} \). Prove the following:

(a) \( N + 1 \) is an absorbing state.

(b) If \( 0 < p < 1 \), then \( h_i^A(p) = 1 \) for all \( i = 1, 2, \ldots, N + 1 \).

(c) If \( p = 0 \), the \( h_i^A(p) = 0 \) for all \( i = 2, \ldots, N \), and \( h_1^A(p) = h_{N+1}^A(p) = 1 \).

(d*) Compute \( k_i^A(p) = E_i(H^A) \) for all \( i \) and all \( 0 \leq p \leq 1 \). For which \( i \) is this lowest and highest? Does this make sense?

(e*) Show that \( k_i^A(\cdot) \) is discontinuous at 0, i.e.,

\[
k_1^A(0) \neq \lim_{p \downarrow 0} k_i^A(p).
\]

Explain this paradox.

**Solution:**

(a) Clearly, \( \{N + 1\} \) is a communicating class, and it is closed.

(b) For simplicity, we will not write the superscript \( A \) unless needed. Fix \( p \in (0, 1) \) and \( i \in \{2, 3, \ldots, N - 1\} \). Then we have

\[
h_i = ph_{i+1} + (1 - p)h_i
\]

or

\[
ph_i = ph_{i+1}.
\]

Since \( p > 0 \), this means that \( h_i = h_{i+1} \), so that \( h_2 = h_3 = \cdots = h_N \). We also have

\[
h_N = ph_1 + (1 - p)h_N \text{ or } ph_1 = ph_N \text{ or } h_1 = h_N.
\]

Finally, note that

\[
h_1 = ph_2 + (1 - p)h_{N+1} = ph_2 + (1 - p)
\]

as \( h_{N+1} = 1 \). Using the fact that \( h_1 = h_2 \) and \( p < 1 \), we obtain \( h_1 = 1 \), and therefore \( h_i = 1 \) for all \( i = 1, 2, \ldots, N \).
(c) If \( p = 0 \), we can solve the Markov chain exactly. If \( X_0 = N + 1 \), then \( X_n = N + 1 \) for all \( n \geq 1 \). If \( X_0 = 1 \), then \( X_n = N + 1 \) for all \( n \geq 1 \). If \( X_0 = i \) for some \( i \in \{2, 3, \ldots, N\} \), then \( X_n = i \) for all \( n \geq 1 \). The result follows.

(d) We similarly approach these equations. Clearly, \( k_{N+1} = 0 \) for all \( p \).

For \( p = 0 \), it is easy to see from the previous part that \( k_1 = 1, k_2 = k_3 = \ldots = k_N = \infty, k_{N+1} = 0 \).

For \( p = 1 \), we have \( k_1 = k_2 = k_3 = \ldots = k_N = \infty, k_{N+1} = 0 \).

Thus fix \( p \in (0, 1) \). For \( i = 2, 3, \ldots, N - 1 \), we have

\[
    k_i = pk_{i+1} + (1 - p)k_i + 1,
\]

or \( pk_i = pk_{i+1} + 1 \),

or \( k_i = k_{i+1} + \frac{1}{p} \).

We obtain a similar equation for \( k_N \) and \( k_1 \), so we have \( k_N = k_1 + \frac{1}{p} \) as well. We can thus recursively determine that

\[
    k_2 = k_N + \frac{N - 2}{p} = k_1 + \frac{N - 1}{p}.
\]

We now have that

\[
    k_1 = pk_2 + (1 - p)k_{N+1} + 1,
\]

or \( k_1 = pk_2 + 1 = pk_1 + (N - 1) + 1 = pk_1 + N, \)

or \( (1 - p)k_1 = N \).

Thus when \( 0 < p < 1 \), we have

\[
    k_1 = \frac{N}{1 - p}, \quad k_i = k_1 + \frac{N + 1 - i}{p} \quad \text{for} \ i = 2, 3, \ldots, N.
\]

We see from these formulas that \( k_1 \) is the lowest (except of course for \( k_{N+1} = 0 \) ), and this makes sense, because all paths to \( N + 1 \) lead through state 1. Similarly, we see that \( k_2 \) is the highest, and again this makes sense from the topology: once we are at state 2, we must go through the entire sequence of states to wrap around and get back to 1.

(e) For \( p \in (0, 1) \) we have

\[
    k_i^A(p) = \frac{N}{1 - p}, \quad \text{thus} \quad \lim_{p \to 0^+} k_i^A(p) = N,
\]

whereas \( k_1^A(0) = 1 \). Thus \( k_i^A(\cdot) \) is discontinuous at 0. This can be explained by the discontinuity in the Markov chain at \( p = 0 \). For \( p > 0 \), \( 1 \rightarrow i \rightarrow 1 \) for every \( i = 2, 3, \ldots, N \), whereas for \( p = 0 \) we have \( 1 \not\rightarrow i \not\rightarrow 1 \) for every \( i = 2, 3, \ldots, N \) and \( 1 \rightarrow N + 1 \) in one step with probability 1. When \( p > 0 \), starting from 1, the chain will come back to 1 in finite steps (with mean \( 1 + (N - 1)/p \) as the time spent at state \( i \) is geometric with mean \( 1/p \) for \( i = 2, 3, \ldots, N \) and 1 step to go from \( N \) to 1) without visiting \( N + 1 \) with probability \( p > 0 \). This number of loops at 1 before visiting \( N + 1 \) is geometric with mean \( p/(1 - p) \). The total amount of time spent at the loop \( 1 \rightarrow 1 \) has mean \( (N - 1 + p)/p \cdot p/(1 - p) = (N - 1 + p)/(1 - p) = N/(1 - p) - 1 \), hence explains the formula for \( k_i^A(p) \) for \( p \in (0, 1) \). For \( p = 0 \), the loops are all missing, creating discontinuity.

4. (1+2+1 pts.) Consider the transition matrix

\[
    P = \frac{1}{2} \begin{bmatrix}
        0 & 1 & 0 & 1 \\
        1 & 0 & 1 & 0 \\
        0 & 1 & 0 & 1 \\
        1 & 0 & 1 & 0
    \end{bmatrix}
\]

(a) Draw the graph corresponding to the Markov chain.

(b) Compute \( P_i(X_n = j) \) for all \( i, j \in \{1, 2, 3, 4\} \) and \( n \geq 0 \).

(c) Describe in words what happens to the stochastic process over a long time.
5. (2+1 pts.) (a) Let $i, j, k \in I$ and $m, n \geq 0$. Show that

$$p_{ij}^{(m+n)} \geq p_{ik}^{(m)} p_{kj}^{(n)}$$

where

$$p_{ij}^{(n)} = (P^n)_{ij}.$$ 

(b) Under what conditions are they equal?

**Solution:** Draw the Picture!

It is easily to check that

$$P^2 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and $P^3 = P$. Thus $P^{2n+1} = P$ and $P^{2n} = P^2$ for all $n \geq 1$. Hence the chain oscillates between the two classes $\{1, 3\}$ and $\{2, 4\}$ with $P_1(X_n = j) = 1/2$ if $n > 0$ is even and $i, j$ in the same class; or $n$ is odd and $i, j$ in different classes; and is zero otherwise.

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6. (3+2* pts.) Consider the process where we only observe the Markov chain when it moves. Let $(X_t)_{t \geq 0} \sim \text{Markov}(\lambda, P)$, where we assume that $p_{ii} < 1$ for all $i \in I$. Define the random times

$$S_0 = 0, S_{m+1} = \inf\{t > S_m \mid X_t \neq X_{S_m}\},$$

and define $Z_m = X_{S_m}, m \geq 0$.

(a) Show that $(Z_m)_{m \geq 0}$ is a Markov chain.

(b) Compute its transition matrix.

**Solution:** (a) First, we note that the random variables $(S_m)_{m \geq 0}$ are stopping times. This follows by induction as $S_0 \equiv 0$ is a constant, hence a stopping time and

$$\{S_{m+1} = k\} = \bigcup_{i=0}^{k-1} \{S_{m-1} = i, X_1 = X_{i+1} = \cdots = X_{k-1} \neq X_k\}.$$ 

The rest of the proof follows similar to the proof for the consecutive hitting times.

(b) We have

$$P(X_{S_1} = y \mid X_{S_0} = x) = \sum_{k=1}^{\infty} P(S_1 = k, X_k = y \mid X_0 = x) = \sum_{k=1}^{\infty} p_{x}^{k-1} p_{xy} = \frac{p_{xy}}{1 - p_{xx}}.$$
for $x \neq y \in I$.

(a) Show that every transition matrix on a finite state-space has at least one closed communicating class.

(b) Find an example of a transition matrix with no closed communicating class.

**Solution:** Recall that a communicating class $C$ is closed if $i \rightarrow j, i \in C$ implies $j \in C$. Moreover, if $i \rightarrow j$ and $i, k$ are in the same communicating class, then $k \rightarrow j$.

Now the state space is finite, hence there are finitely many communicating classes, say $C_1, C_2, \ldots, C_n$. Assume that none of them are closed. Let $\ell_1 = 1$. Since $C_{\ell_1} = C_1$ is not closed, there exists $i_1 \in C_{\ell_1}$ and $i_2 \in C_{\ell_2}, \ell_2 \neq 1$, such that $i_1 \rightarrow i_2$ (that is, the chain escapes from $C_{\ell_1}$ to some other class $C_{\ell_2}$). Since the communicating class $C_{\ell_2}$ is not closed, there exists $i_3 \in C_{\ell_3}, \ell_3 \neq \ell_2$, such that $i_2 \rightarrow i_3$. By continuing this process we obtain a sequence $(C_{\ell_k})_{k \geq 1}$ of communicating classes and a sequences of states, $(i_k)_{k \geq 1}$ such that $i_k \in C_{\ell_k}$ and $i_k \rightarrow i_{k+1}$ for all $k \geq 1$. Since the number of communicating classes is finite, there must be at least one class that appears at least twice in the sequence. Without loss of generality, we assume that the class $C_1$ appears twice. Thus we obtain that

$$i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_m \in C_1$$

for some $m > 2$. Since $i_m \rightarrow i_1$ (they are both in the same communicating class), we close the circle. This means that all states in the above sequence communicate. This is a contradiction with the fact that at least $C_{\ell_2}$ is different from $C_1$!

For the example, consider the Markov Chain with state space $I = \{1, 2, \ldots\}$ with transition matrix $P = ((p_{ij}))_{i,j \geq 1}$ where $p_{i,i+1} = 1$ and 0 otherwise. Clearly, there are no closed communicating classes and $i \rightarrow i + 1 \not\rightarrow i$ for all states $i$. 