Homework 2
MATH 564/STAT 555/MATH 466: Applied Stochastic Processes
Due date: September 22, 2022

For Math 466, skip questions marked with ∗; the total point is 20. For Math 564/Stat 555, answer all questions; the total point is 30. Write your name and course number (i.e., Math 564, Math 466, Stat 555) clearly on top of the first page. You can work in groups or on your own. Please indicate whom you worked with; it will not affect your grade in any way.

1. (3 pts.) Suppose that \((X_n)_{n \geq 0}\) is Markov(\(\lambda, P\)). Define \(Y_n = X_{kn}\) for some \(k \geq 1\). Show that \((Y_n)_{n \geq 0}\) is also a Markov chain. Find the initial distribution and transition matrix.

2. (3 pts.) Show that for any \(i, j \in I, A \subseteq I\) and \(i \notin A\) where \(I\) is the state space, we have
\[
P_i(H_A < \infty \mid X_1 = j) = P_j(H_A < \infty)
\]
and
\[
E_i(H_A \mid X_1 = j) = 1 + E_j(H_A).
\]

3. (1+2+1+3∗+2∗ pts.) We consider a Markov chain which is given by a unidirectional ring with one escape point, i.e., pick an integer \(N > 1\) and \(0 \leq p \leq 1\), and consider the Markov chain with state space \(I = \{1, 2, \ldots, N + 1\}\) and transition probabilities
\[
p_{1,1} = 0, \quad p_{1,2} = p, \quad p_{1,N+1} = 1 - p,
p_{i,i+1} = p, \quad p_{i,i} = 1 - p, \quad \text{for } i = 2, 3, \ldots, N - 1
\]
\[
p_{N,1} = p, \quad p_{N,N} = 1 - p,
p_{N+1,N+1} = 1.
\]
For \(N = 4\), the corresponding graph is:

Denote \(A = \{N + 1\}\). Prove the following:

(a) \(N + 1\) is an absorbing state.
(b) If \(0 < p < 1\), then \(h^A_i(p) = 1\) for all \(i = 1, 2, \ldots, N + 1\).
(c) If \(p = 0\), the \(h^A_i(p) = 0\) for all \(i = 2, \ldots, N\), and \(h^A_i(p) = h^A_{N+1}(p) = 1\).
(d∗) Compute \(k^A_i(p) = E_i(H^A)\) for all \(i\) and all \(0 \leq p \leq 1\). For which \(i\) is this lowest and highest? Does this make sense?
(e∗) Show that \(k^A_i(\cdot)\) is discontinuous at 0, i.e.,
\[
k^A_i(0) \neq \lim_{p \downarrow 0} k^A_i(p).
\]
Explain this paradox.
4. (1+2+1 pts.) Consider the transition matrix

\[ P = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \]

(a) Draw the graph corresponding to the Markov chain.
(b) Compute \( P_i(X_n = j) \) for all \( i, j \in \{1, 2, 3, 4\} \) and \( n \geq 0 \).
(c) Describe in words what happens to the stochastic process over a long time.

5. (2+1 pts.) (a) Let \( i, j, k \in I \) and \( m, n \geq 0 \). Show that

\[ p_{ij}^{(m+n)} \geq p_{ik}^{(m)} p_{kj}^{(n)} \]

where

\[ p_{ij}^{(n)} = (P^n)_{ij}. \]

(b) Under what conditions are they equal?

6. (3+2* pts.) Consider the process where we only observe the Markov chain when it moves. Let \( (X_t)_{t \geq 0} \sim \text{Markov}(\lambda, P) \), where we assume that \( p_{ii} < 1 \) for all \( i \in I \). Define the random times

\[ S_0 = 0, S_{m+1} = \inf\{t > S_m \mid X_t \neq X_{S_m}\}, \]

and define \( Z_m = X_{S_m}, m \geq 0 \).
(a) Show that \( (Z_m)_{m \geq 0} \) is a Markov chain.
(b) Compute its transition matrix.

7*. (2*+1* pts.) (a) Show that every transition matrix on a finite state-space has at least one closed communicating class.
(b) Find an example of a transition matrix with no closed communicating class.