Homework 1

MATH 564/466/STAT 555: Applied Stochastic Processes

Due date: Sep 8, 2022

Answer all questions. Write your name and course number (i.e., Math 564, Math 466, Stat 555) clearly on top of the first page. Please indicate whom you worked with, it will not affect your grade in any way.

1. (2 pts.) Prove that, \( \lim \sup A_n = \{ x : x \text{ is in infinitely many } A_n \} \).

2. (2+2 pts.) (a) Prove the Law of Total Probability, i.e., if \( \{B_n\} \) is a partition of \( \Omega \), then for an event \( A \) we have
   \[
   P(A) = \sum P(A | B_n) P(B_n).
   \]
   (b) Deduce that if \( X \) and \( Y \) are discrete random variables then the following are equivalent: (i) \( X \) and \( Y \) are independent, i.e., \( P(X = x, Y = y) = P(X = x) P(Y = y) \) for all \( x, y \); (ii) the conditional distribution of \( X \) given \( Y = y \) is independent of \( y \).

3. (2 pts.) Prove the Inclusion-Exclusion Identity for two events:
   \[
P(A \cup B) = P(A) + P(B) - P(A \cap B).
   \]

4. (2+2+2 pts.) We define the function \( \phi : \mathbb{N} \to \mathbb{N} \) as the number of distinct \( \sigma \)-algebras defined on the set \( S = \{1, 2, 3, \ldots, n\} \).
   (a) Let \( \psi(n) \) be the number of distinct collections of subsets of \( S \). Show that \( \psi(n) = 2^{2^n} \) and that \( \phi(n) \leq \psi(n) \).
   (b) Show that, for \( n > 1 \), \( \phi(n + 1) > \phi(n) \) so that \( \phi \) is strictly increasing function.
   (c) Compute by hand \( \phi(k) \) for \( k = 0, 1, 2, 3, 4 \) by enumerating all possible \( \sigma \)-algebras (of course, the theorem about the correspondence between sigma algebras and partitions will help). Do you see a pattern? Perhaps you might find this pattern in the Integer Sequence Database. How does the growth of this sequence compare to the bounds in part (a)?

5. (2 pts.) Let \( X \) be a random variable that takes values in \( \mathbb{N} := \{0, 1, 2, \ldots\} \). Prove that
   \[
   E(X) = \sum_{n=1}^{\infty} P(X \geq n).
   \]

6. (2+2 pts.) We will flip three fair coins, and use \( \Omega = \{H, T\}^3 \) as our probability space. We define \( X \) as the total number of heads on the three flips, and define \( \mathcal{F}_1 \) as the information gained after one flip.
   (a) Show that \( E(X | \mathcal{F}_1) : \Omega \to \mathbb{R} \) only takes two values, and determine the sets on which they are constant.
   (b) Compute these two values using the theorem about linearity of expectation, by writing everything out explicitly.

7. (4 pts.) Given an edge-weighted directed graph \( G = (V, E) \) with edge weights \( a_{ij} \) for an edge \( (i, j) \in E \), define the weight matrix \( A = ((a_{ij}))_{i,j\in V} \), where \( a_{ij} = 0 \) if \( (i, j) \notin E \). Prove that, for any integer \( k \geq 0 \) the total weight of all paths of length \( k \) starting at vertex \( i \) and ending at vertex \( j \) is given by the \((i, j)\)-th entry of the matrix \( A^k \).

8. (3+3 pts.) Let \( \Omega \) be a countable set and \( \mathcal{F} \) be a \( \sigma \)-algebra on \( \Omega \). Define the relation \( \sim \) on \( \Omega \) as follows: \( x \sim y \) if \( x \in A \) iff \( y \in A \) for all \( A \in \mathcal{F} \). Clearly \( \sim \) is an equivalence relation.
   (a) Let \( (E_i, i \geq 1) \) be the disjoint equivalence classes w.r.t. \( \sim \) giving a partition of \( \Omega \). Show that \( E_i \in \mathcal{F} \) for all \( i \) and thus \( \sigma(E_i, i \geq 1) \subseteq \mathcal{F} \).
   (b) Show that
   \[
   \mathcal{F} \subseteq \sigma(E_i, i \geq 1).
   \]