A financial derivative is a contract whose value depends on one or more securities or assets, called **underlying assets**. Typically the underlying asset is a stock, a bond, a currency exchange rate or the quotation of commodities such as gold or oil.

### 29.1 Option

An option is the simplest example of a derivative instrument. An option is a contract that gives the right (but not the obligation) to its holder to buy or sell some amount of the underlying asset at a future date, for a prespecified price. Therefore in an option contract we need to specify:

(a) an underlying asset;

(b) an exercise price \( K \), the so-called strike price;

(c) a date \( T \), the so-called maturity.

#### 29.1.1 Call and Put options

A Call option gives the right to buy, whilst a Put option gives the right to sell. An option is called European if the right to buy or sell can be exercised only at maturity, and it is called American if it can be exercised at any time before maturity.

Let us consider a European Call option with strike \( K \), maturity \( T \) (years) and let us denote the price of the underlying asset at maturity by \( S_T \). At time \( T \) we have two possibilities: if \( S_T > K \), the payoff of the option is equal to \( S_T - K \), corresponding to the profit obtained by exercising the option (i.e. by buying the underlying asset at price \( K \) and then selling it at the market price \( S_T \)). If \( S_T < K \), exercising the option is not profitable and the payoff is zero. In conclusion the payoff of a European Call option is

\[
(S_T - K)^+ = \max\{S_T - K, 0\}.
\]

Note that the payoff increases with \( S_T \) and gives a potentially unlimited profit. Analogously, we see that the payoff of a European Put option is

\[
(K - S_T)^+ = \max\{K - S_T, 0\}.
\]

In general, we will consider European option with payoff function \( \phi \) at expiry \( T \); i.e., we get the payoff \( \phi(S_T) \) at time \( T \).
### 29.1.2 Compound Interest

It is good to recall some notions on the time value of money in finance: receiving $1 today is not like receiving it after a month. The rules of compounding express the dynamics of an investment with fixed risk-free interest rate: to put it simply, this corresponds to deposit the money on a savings account. In the financial modeling, it is always assumed that a (locally) risk-free asset, the so-called bond, exists. If $D_t$ is the value of the bond at time $t$, the formula of continuous compounding with annual interest rate $r$ is given by

$$D_T = \lim_{n \to \infty} D_0 (1 + rT/n)^n = D_0 e^{rT}.$$ 

Since to obtain a final wealth (at time $T$) equal to $D$, it is necessary to invest the amount $De^{-rT}$ at the initial time, this amount is usually called discounted value of $D$.

### 29.1.3 Arbitrage Opportunities

Broadly speaking an arbitrage opportunity is the possibility of carrying out a financial operation without any investment, but leading to profit without any risk of a loss. From a theoretical point of view it is evident that a sensible market model must avoid this type of profit. As a matter of fact, the no-arbitrage principle has become one of the main criteria to price financial derivatives.

The idea on which arbitrage pricing is built is that, if two financial instruments will certainly have the same value at future date, then also in this moment they must have the same value. If this were not the case, an obvious arbitrage opportunity would arise: by selling the instrument that is more expensive and by buying the less expensive one, we would have an immediate risk-free profit since the selling position (short position) on the more expensive asset is going to cancel out the buying position (long position) on the cheaper asset. Concisely, we can express the no-arbitrage principle in the following way:

$$X_T \leq Y_T \implies X_t \leq Y_t \text{ for all } t \leq T,$$

where $X_t$ and $Y_t$ are the values of the two financial instruments respectively.

### 29.2 Pricing

We consider a financial market that is free from arbitrage opportunities. In the Black-Scholes model the market consists of a non-risky asset, a bond $D$ and of a risky asset, a stock $S$. The bond price verifies the equation

$$dD_t = rD_t dt$$

where $r$ is the short-term (or locally risk-free) interest rate, assumed to be a constant. The price of the risky asset is a geometric Brownian motion, verifying the equation

$$dS_t = bS_t dt + \sigma S_t dB_t$$

where $(B_t)_{t \geq 0}$ is a SBM, $b$ is the average rate of return and $\sigma$ is the volatility. Recall that the explicit expression for $D_t, S_t$ are given by

$$D_t = D_0 e^{rt}, \quad S_t = S_0 e^{\sigma B_t + (b - \sigma^2/2)t}.$$
An option is a contract whose final value is given, this depending on the price of the underlying asset at maturity which is not known at present. Therefore the non-trivial problem of pricing arises, i.e., the determination of the “rational” or fair price of the option: this price is the premium that the buyer of the option has to pay at the initial time to get the right guaranteed by the contract.

In this class, we consider pricing European option with payoff function $\phi$ at expiry $T$; i.e., we get the payoff $\phi(S_T)$ at time $T$. We assume that the current interest rate is $r$.

### 29.2.1 Self-financing strategies

**Definition 29.1.** A strategy (or portfolio) is a stochastic process $(\alpha_t, \beta_t)$ where $\alpha \in L^2_{\text{loc}}$ and $\beta \in L^1_{\text{loc}}$. The value of the portfolio $(\alpha, \beta)$ is the stochastic process defined by

$$V_t^{(\alpha, \beta)} = \alpha_t S_t + \beta_t D_t.$$  

The processes $\alpha, \beta$ are to be interpreted as the amount of $S$ and $D$ held by the investor in the portfolio: let us point out that short-selling is allowed, so $\alpha, \beta$ can take negative values. Where there is no risk of ambiguity, we simply write $V$ instead of $V^{(\alpha, \beta)}$.

Intuitively the assumption that $\alpha, \beta$ have to be progressively measurable describes the fact that the investment strategy depends only on the amount of information available at that moment.

**Definition 29.2.** A strategy $(\alpha_t, \beta_t)$ is self-financing if

$$dV_t^{(\alpha, \beta)} = \alpha_t dS_t + \beta_t dD_t$$ holds, i.e.,

$$V_t^{(\alpha, \beta)} = V_0^{(\alpha, \beta)} + \int_0^t \alpha_s dS_s + \int_0^t \beta_s dD_s.$$  

From a purely intuitive point of view, this expresses the fact that the instantaneous variation of the value of the portfolio is caused uniquely by the changes of the prices of the assets, and not by injecting or withdrawing funds from outside.

### 29.2.2 Markovian Strategies and Black-Scholes equation

**Definition 29.3.** A strategy $(\alpha_t, \beta_t)$ is Markovian if

$$\alpha_t = \alpha(t, S_t), \quad \beta_t = \beta(t, S_t)$$  

where $\alpha, \beta$ are functions in $C^{1,2}([0, T] \times (0, \infty))$.

The value of a Markovian strategy $(\alpha, \beta)$ is a function of time and of the price of the underlying asset:

$$f(t, S_t) := V_t^{(\alpha, \beta)} = \alpha(t, S_t) S_t + \beta(t, S_t) D_0 e^{rt}.$$  

By the fact that $S_t$ has a strictly positive (log-normal) density on $(0, \infty)$, one can show that the function $f$ is uniquely determined by $(\alpha, \beta)$. The following result characterizes the self-financing condition of a Markovian portfolio in differential terms.
**Theorem 29.4.** Suppose that \((\alpha, \beta)\) is a Markovian strategy and set \(f(t, S_t) := V_t^{(\alpha, \beta)}\). The following two conditions are equivalent:

(i) \((\alpha, \beta)\) is self-financing.

(ii) \(f\) is solution to the partial differential equation

\[
\frac{\sigma^2 x^2}{2}\partial_{xx} f(t, x) + r x \partial_x f(t, x) + \partial_t f(t, x) = rf(t, x) \tag{29.1}
\]

with \((t, x) \in [0, T) \times (0, \infty)\) and we have that

\[
\alpha(t, x) = \partial_x f(t, x), \quad \beta(t, x) = \frac{f(t, x) - \alpha(t, x)x}{D_t}.
\]

Equation (29.1) is called Black-Scholes partial differential equation.

**Proof.** \([i) \implies ii)\] By the self-financing condition and expression for \(S\), we have that

\[
dV_t^{(\alpha, \beta)} = (\alpha_t b S_t + \beta_t r D_t) dt + \alpha_t \sigma S_t dB_t.
\]

Then, by the Itô’s formula and putting for brevity \(f = f(t, S_t)\), we have that

\[
dV_t^{(\alpha, \beta)} = \partial_t f dt + \partial_x f dS_t + \frac{1}{2} \partial_{xx} f d\langle S \rangle_t
\]

\[
= (\partial_t f + b S t \partial_x f + \frac{1}{2} \sigma^2 S_t^2 \partial_{xx} f) dt + \sigma S_t \partial_x f dB_t.
\]

From the uniqueness of the representation of an Itô process, we get

\[
\alpha_t = \partial_x f(t, S_t) \text{ a.s.}
\]

Comparing the terms in \(dt\), we get

\[
\partial_t f(t, S_t) + b S_t \partial_x f(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \partial_{xx} f(t, S_t) = \alpha_t b S_t + \beta_t r D_t = b S_t \partial_x f(t, S_t) + r (f - S_t \partial_x f(t, S_t))
\]

or

\[
\frac{1}{2} \sigma^2 S_t^2 \partial_{xx} f(t, S_t) + r S_t \partial_x f(t, S_t) + \partial_t f(t, S_t) = rf(t, S_t).
\]

Here we used the fact that \(\beta_t D_t = f(t, S_t) - \alpha_t S_t\). Hence we have the result.

The other direction is simply an application of Itô’s formula. \(\blacksquare\)

One can also prove the following result.

**Theorem 29.5.** Let

\[
c = \frac{r}{\sigma} - \frac{\sigma^2}{2}, \quad d = r + \frac{\sigma^2}{2}.
\]

Then the function \(f\) is a solution of the Black-Scholes equation iff the function

\[
u(t, x) := e^{xc+td} f(T - t, e^{\sigma x}), t \in [0, T], x \in \mathbb{R}
\]

satisfies the heat equation

\[
\frac{1}{2} \partial_{xx} u = \partial_t u \text{ in } (0, T) \times \mathbb{R}.
\]
Definition 29.6. A strategy is admissible if it is Markovian, self-financing, bounded from below. An European derivative $\phi(S_T)$ is replicable if there exists an admissible portfolio $(\alpha, \beta)$ such that

$$V_T^{(\alpha, \beta)} = \phi(S_T).$$

We say that $(\alpha, \beta)$ is a replicating portfolio for $\phi(S_T)$.

The bounded from below assumption makes sure that unlimited debt are not allowed. The following theorem is the central result in Black-Scholes theory and gives the definition of arbitrage price of a derivative.

Theorem 29.7. The Black-Scholes market model is complete and arbitrage-free, i.e., every European derivative $\phi(S_T)$, with $\phi$ being lower bounded and “moderately growing”, is replicable in a unique way. Indeed there exists a unique admissible strategy $(\alpha_t, \beta_t)$ replicating $\phi(S_T)$, that is given by

$$\alpha_t = \partial_x f(t, S_t), \quad \beta_t = \frac{f(t, S_t) - S_t \partial_x f(t, S_t)}{D_t},$$

where $f$ is the lower bounded solution of the Cauchy problem

$$\frac{\sigma^2 x^2}{2} \partial_{xx} f(t, x) + r x \partial_x f(t, x) + \partial_t f(t, x) = r f(t, x) \quad \text{in} \ [0, T) \times (0, \infty)$$

$$f(T, x) = \phi(x) \quad \text{for} \ x > 0.$$

By definition, $f(t, S_t) = V_T^{(\alpha_t, \beta_t)}$ is the arbitrage price for $\phi(S_T)$.

The “moderate growth” condition can be written as $\phi(x) \leq ce^{c|\log x|^{2-\varepsilon}}$, $c > 0$, $\varepsilon \in (0, 1)$. Thus, the fair price for the option at time 0 is $f(0, S_0)$. Using Feynmann-Kac formula we can write $f(0, x)$ as the expectation $E(e^{-rT} \phi(x e^{\sigma B_T + (r - \sigma^2/2) T}))$.