26.1 Feller Semigroups

To define some continuity properties of $Q_t$ w.r.t. time $t$, we need a norm or way to measure distance between functions.

Assume that $E$ is a metrizable locally compact topological space. We also assume that $E$ is countable at infinity, meaning that $E$ is a countable union of compact sets. The space $E$ is equipped with its Borel $\sigma$-field and is Polish. Let $C_0(E)$ denote the set of all continuous real-valued functions on $E$ that vanishes at infinity (or uniformly smaller than arbitrary $\varepsilon > 0$ outside some compact set). The space $C_0(E)$ is a Banach space for the supremum norm.

**Definition 26.1 (Feller semigroup).** Let $(Q_t)_{t \geq 0}$ be a transition semigroup on $E$. We say that $(Q_t)_{t \geq 0}$ is a *Feller semigroup* if the following holds:

1. $Q_t f \in C_0(E)$ for every $f \in C_0(E)$,

2. $\lim_{t \downarrow 0} \|Q_t f - f\| = 0$ for every $f \in C_0(E)$.

A Markov process with values in $E$ is a *Feller process* if its semigroup is Feller.

One can prove that condition 2. can be replaced by the seemingly weaker property

$$\lim_{t \downarrow 0} Q_t f(x) = f(x) \text{ for all } f \in C_0(E), x \in E.$$ 

Condition 2. also implies that, for every $s \geq 0$,

$$\lim_{t \downarrow 0} \|Q_{t+s} f - Q_s f\| = \lim_{t \downarrow 0} \|Q_s(Q_t f - f)\| = 0$$

since $Q_s$ is a contraction on $C_0(E)$. Note that the convergence is uniform when $s$ varies over $\mathbb{R}_+$, which ensures that the mapping $t \mapsto Q_t f$ is uniformly continuous from $\mathbb{R}_+$ into $C_0(E)$, for any fixed $f \in C_0(E)$.

We fix a Feller semigroup $(Q_t)_{t \geq 0}$ on $E$. Using property 1. of the definition and the DCT, it easily follows that $R_\lambda f \in C_0(E)$ for every $f \in C_0(E)$ and $\lambda > 0$. We will restrict all the functions to $C_0(E)$.

**Lemma 26.2.** Let $\lambda > 0$ and set $\mathcal{R} = \text{Range}(R_\lambda) = \{R_\lambda f \mid f \in C_0(E)\}$. Then $\mathcal{R}$ does not depend on the choice $\lambda > 0$ and is a dense subspace of $C_0(E)$.

**Proof.** For $\lambda \neq \mu$, the resolvent equation gives $R_\lambda f = R_\mu(f + (\mu - \lambda)R_\mu f)$. Hence any function of the form $R_\lambda f$ with $f \in C_0E$ is also of the form $R_\mu g$ for some $g \in C_0E$. This proves the first statement.
Clearly, $\mathcal{R}$ is a linear subspace of $C_0(E)$. For the second statement we note that with $\tau \sim \text{Exponential}(1)$ we have
\[
\lim_{\lambda \to \infty} \|\lambda R_\lambda f - f\| = \lim_{\lambda \to \infty} \|E(Q_{\tau/\lambda} - Q_0)f\| \leq \lim_{\lambda \to \infty} E \|Q_{\tau/\lambda} - Q_0)f\| = 0
\]
where the last line follows by Feller property and DCT. ■

**Definition 26.3.** We define
\[
D := \{ f \in C_0(E) \mid \text{there exists } g \in C_0(E) \text{ such that } \lim_{t \downarrow 0} \| (Q_t f - f)/t - g \| = 0 \}
\]
and for every $f \in D$,
\[
Lf := \lim_{t \downarrow 0} \frac{1}{t}(Q_t f - f).
\]

Then $D$ is a linear subspace of $C_0(E)$ and $L : D \to C_0(E)$ is a linear operator called the generator of the semigroup $(Q_t)_{t \geq 0}$. The subspace $D$ is called the domain of $L$ and will be denoted by $D(L)$, when needed. Note that $Q_{s+t}f - Q_s f = Q_s(Q_tf - f)$. Thus we have the following.

**Proposition 26.4.** Let $f \in D$ and $s > 0$. Then $Q_s f \in D$ and $L(Q_s f) = Q_s(Lf)$.

Moreover, by uniform continuity of $t \mapsto Q_t f$ and fundamental theorem of calculus, we get the following.

**Proposition 26.5.** for $f \in D$ and $t \geq 0$, we have
\[
Q_t f = f + \int_0^t Q_s(Lf) \, ds.
\]

The next proposition identifies the domain $D$ in terms of the resolvent operators $R_\lambda$.

**Proposition 26.6.** Let $\lambda > 0$. We have
1. for every $g \in C_0(E)$, $R_\lambda g \in D$ and $(\lambda - L)R_\lambda g = g$;
2. for $f \in D$, $R_\lambda(\lambda - L)f = f$.

Consequently, $D = \mathcal{R}$ and the operators $R_\lambda : C_0(E) \to \mathcal{R}$ and $\lambda - L : \mathcal{R} \to C_0(E)$ are the inverse of each other.

**Proof.** 1. If $g \in C_0(E)$, we have for every $\varepsilon > 0$
\[
\varepsilon^{-1}(Q_\varepsilon R_\lambda g - R_\lambda g) = \varepsilon^{-1} \left( \int_0^\infty e^{-\lambda t} Q_{\varepsilon+t} g dt - \int_0^\infty e^{-\lambda t} Q_t g dt \right)
\]
\[
= \varepsilon^{-1} (1 - e^{-\lambda \varepsilon}) \int_0^\infty e^{-\lambda t} Q_{\varepsilon+t} g dt - \varepsilon^{-1} \int_0^\varepsilon e^{-\lambda t} Q_t g dt
\]
\[
\to \lambda R_\lambda g - g
\]
in $C_0(E)$ as $\varepsilon \to 0$ using Feller property to get continuity. Thus $R_\lambda g \in D$ and $LR_\lambda g = \lambda R_\lambda g - g$ or $(\lambda - L)R_\lambda g = g$. 
2. Let $f \in D$. Using $Q_tf = f + \int_0^t Q_s(Lf) \, ds$ we have

$$\lambda R_\lambda f = \int_0^\infty \lambda e^{-\lambda t} Q_t f \, dt = f + \int_0^\infty \lambda e^{-\lambda s} Q_s(Lf) \, ds = f + R_\lambda(Lf).$$

We thus have the result. □

As a corollary we get that

**Corollary 26.7.** The semigroup $(Q_t)_{t \geq 0}$ is determined by the generator $L$ and the domain $D$.

Now we look at the Standard Brownian Motion where $Q_tf(x) = E f(x+\sqrt{t}\eta), \eta \sim N(0,1)$ which can be easily verified to be a Feller process by directly checking the Feller conditions. Recall that, with $\xi$ having density $2^{-1/2}e^{-\sqrt{2}|y|}dy, y \in \mathbb{R}$

$$R_\lambda f(x) = \lambda^{-1} E f(x + \lambda^{-1/2} \cdot \xi) = (2\lambda)^{-1/2} \int_R f(y)e^{-\sqrt{2}\lambda|y-x|} \, dy.$$ 

Taking $\lambda = 1/2$, we get $R_{1/2}f(x) = \int_R f(y)e^{-|y-x|} \, dy$. If $h \in D$, we know that there exists an $f \in C_0(\mathbb{R})$ such that

$$h(x) = R_{1/2}f(x) = \int_R f(y)e^{-|y-x|} \, dy.$$ 

By differentiating under the integral sign, we have

$$h'(x) = \int_R \text{sgn}(y-x)f(y)e^{-|y-x|} \, dy.$$ 

We claim that $h'$ must also be differentiable on $\mathbb{R}$. For $x > x_0$, we have

$$h'(x) - h'(x_0) = - \int_{x_0}^x (e^{-|y-x|} + e^{-|y-x_0|}) f(y) \, dy$$ 

$$+ \int_{[x_0,x]\cap \mathbb{R}} \text{sgn}(y-x_0)(e^{-|y-x|} - e^{-|y-x_0|}) f(y) \, dy$$

and it follows that

$$\frac{h'(x) - h'(x_0)}{x-x_0} \to -2f(x_0) + h(x_0)$$

as $x \downarrow x_0$. We get the same limit when $x \uparrow x_0$ and thus $h$ is twice differentiable with $h'' = -2f + h$. On the other hand, since $h = R_{1/2}f$, we have $2f = 2(1/2 - L)R_{1/2}f = (1-2L)h$. Hence $Lh = \frac{1}{2}h''$.

Summarizing, we get that

$$D \subseteq \{ h \in C^2(\mathbb{R}) \mid h, h'' \in C_0(\mathbb{R}) \}$$

and if $h \in D$, $Lh = \frac{1}{2}h''$. In fact, the preceding inclusion is an equality and can be proved easily.

In general, it is very difficult to determine the exact domain of the generator. SBM being a special case. The next theorem often allows one to identify elements of this domain using martingales associated with the Markov process with semigroup $(Q_t)_{t \geq 0}$.