Theorem 18.1 (Burkholder-Davis-Gundy Inequality). Let $M$ be a continuous local martingale with $M_0 = 0$. Then for every stopping time $T$ and $p > 0$, there exists constants $c_p$ and $C_p$ such that
\[ c_p E\left[ \left( \sqrt{\langle M \rangle_T} \right)^p \right] \leq E\left[ \left( \sup_{t \leq T} |M_t| \right)^p \right] \leq C_p E\left[ \left( \sqrt{\langle M \rangle_T} \right)^p \right]. \]

Definition 18.2. $M_T^* = \sup_{t \leq T} |M_t|.$

Proof. Replacing $M$ by the stopping martingale $M_T$, we see that it is enough to treat the special case $T = \infty$. We then observe that it suffices to consider the case when $M$ is bounded: Assuming that the bounded case has been treated, we can replace $M$ by $M_{T_n}$, where $T_n = \inf\{ t \geq 0 : |M_t| = n \}$, and we get the general case by letting $n$ tend to $\infty$.

We first prove the left-hand side inequality assuming the right-hand side one is true for any $p > 0$. We apply Ito’s formula to the function $x^2$:
\[ M_\infty^2 = 2 \int_0^\infty M_t dM_t + \langle M \rangle_\infty. \]
Then, we have
\[ \langle M \rangle_\infty^{p/2} = (M_\infty^2 - 2 \int_0^\infty M_t dM_t)^{p/2} \leq (M_\infty^2 + 2 \int_0^\infty M_t dM_t)^{p/2}. \]
By the fact that $|x + y|^\theta \leq a_\theta (|x|^\theta + |y|^\theta)$, where $a_\theta = 1$ if $\theta \in (0, 1)$ and $a_\theta = 2^{\theta - 1}$ if $\theta \in [1, \infty)$, and taking expectation on both sides, we get
\[ \mathbb{E}[\langle M \rangle_\infty^{p/2}] \leq a_{p/2}(\mathbb{E}[|M_\infty|^p] + \mathbb{E}[\int_0^\infty 2 M_t dM_t |^{p/2}]) \]
\[ \leq a_{p/2}(\mathbb{E}[|M_\infty|^p] + C_{p/2}\mathbb{E}[\int_0^\infty 2 M_t dM_t ]^{p/4}). \]
Now since $\langle \int_0^\infty 2 M_t dM_t \rangle_\infty = \int_0^\infty 4 M_t^2 d\langle M \rangle_t \leq 4(M_\infty^*)^2 \langle M \rangle_\infty$, we have
\[ \mathbb{E}[\langle \int_0^\infty 2 M_t dM_t \rangle_\infty^{p/4}] \leq 4^{p/4} \mathbb{E}[\langle M_\infty^* \rangle^{p/2} \langle M \rangle_\infty^{p/4}]. \]
Let $x = \mathbb{E}[\langle M \rangle_\infty^{p/2}]$ and $y = \mathbb{E}[|M_\infty|^p]$, then we have
\[ x \leq a_{p/2}(y + 2^{p/2}C_{p/2}\sqrt{xy}) \leq K_p(y + \sqrt{xy}), \]
where $K_p = a_{p/2}(1 \lor 2^{p/2}C_{p/2}).$
This is equivalent to
\[
(\sqrt{x})^2 - K_p\sqrt{xy} + K_p^2/4(\sqrt{y})^2 \leq (K_p + K_p^2/4)y,
\]
\[
|\sqrt{x} - K_p/2\sqrt{y}| \leq \sqrt{K_p + K_p^2/4}\sqrt{y},
\]
\[
x \leq (K_p/2 + \sqrt{K_p + K_p^2/4})^2y.
\]

Let \(c_p = 1/(K_p/2 + \sqrt{K_p + K_p^2/4})^2\), we conclude the proof.

Next, we prove the right-hand side inequality.

**Case I:** \(p \geq 2\). \(F(x) = |x|^p\) is twice continuously differentiable with \(F'(x) = px|x|^{p-2}\) and \(F''(x) = p(p-1)|x|^{p-2}\). We apply Itô’s formula to \(F(x)\):
\[
|M_s|^p = \int_0^s pM_t|M_t|^{p-2}dM_t + \frac{1}{2}p(p-1)\int_0^s|M_t|^{p-2}d\langle M \rangle_t.
\]

Take \(s = \infty\) and expectation, we have
\[
\mathbb{E}[|M_\infty|^p] = \frac{1}{2}p(p-1)\mathbb{E}\left[\int_0^\infty|M_t|^{p-2}d\langle M \rangle_t\right]
\leq \frac{1}{2}p(p-1)\mathbb{E}[|M_\infty|^p]\mathbb{E}[\langle M \rangle_\infty]
\leq \frac{1}{2}p(p-1)(\mathbb{E}[|M_\infty|^p])^{1-2/p}(\mathbb{E}[\langle M \rangle_\infty^{p/2})]^{2/p}.
\]

Using Doob’s \(L^p\) maximal inequality, we have
\[
\mathbb{E}[|M_\infty|^p] \leq \left(\frac{p}{p-1}\right)^p\mathbb{E}[|M_\infty|^p]
\leq \frac{p^{p+1}}{2(p-1)^{p-1}}(\mathbb{E}[|M_\infty|^p])^{p-2/p}(\mathbb{E}[\langle M \rangle_\infty^{p/2})]^{2/p}.
\]

Then,
\[
(\mathbb{E}[|M_\infty|^p])^{2/p} \leq \frac{p^{p+1}}{2(p-1)^{p-1}}(\mathbb{E}[\langle M \rangle_\infty^{p/2})]^{2/p},
\]
and letting \(C_p = (\frac{p^{p+1}}{2(p-1)^{p-1}})^{p/2}\), we conclude the proof.

**Case II:** \(0 < p < 2\). We know that \(\mathbb{E}[M_T^2] = \mathbb{E}[\langle M \rangle_T]\) for any stopping time \(T\). Define \(H_x = \inf\{t \geq 0 : |M_t|^2 \geq x\}\) and \(S_x = \inf\{t \geq 0 : \langle M \rangle_t \geq x\}\). Let \(q = p/2 \in (0, 1)\). Then we have
\[
\mathbb{P}((M_\infty^*)^2 > x) \leq \mathbb{P}(|\langle M \rangle_\infty | > x) + \mathbb{P}((M_\infty^*)^2 > x, |\langle M \rangle_\infty | < x).
\]

And
\[
\mathbb{P}((M_\infty^*)^2 > x, |\langle M \rangle_\infty | < x) \leq \mathbb{P}(|\langle M_\infty^* \rangle_{H_x \wedge S_x}| > x)
\leq \frac{1}{x}\mathbb{E}[|\langle M_\infty^* \rangle_{H_x \wedge S_x}|]
= \frac{1}{x}\mathbb{E}[\langle M \rangle_{H_x \wedge S_x}]
\leq \frac{1}{x}\mathbb{E}[|\langle M \rangle_\infty \wedge x|] \leq \frac{1}{x}\mathbb{E}[\langle M \rangle_\infty \mathbb{1}_{\langle M \rangle_\infty < x}] + \mathbb{P}(\langle M \rangle_\infty \geq x).
\]
Thus,
\[ \mathbb{P}((M_\infty^2)^2 > x) \leq 2\mathbb{P}((M_\infty) \geq x) + \frac{1}{x} \mathbb{E}[(M_\infty) \mathbf{1}_{(M_\infty < x)}]. \]

Finally, we have
\[
\mathbb{E}[(M_\infty^*)^{2q}] = \int_0^\infty q x^{q-1} \mathbb{P}((M_\infty^*)^2 > x) dx \\
\leq 2 \int_0^\infty q x^{q-1} \mathbb{P}((M_\infty) \geq x) dx + \mathbb{E}[(\int_0^\infty q x^{q-2} dx)(M_\infty)] \\
= (2 + \frac{q}{q-1}) \mathbb{E}[(M_\infty^*)^q],
\]
which concludes the proof.

**Corollary 18.3.** Let \( M \) be a continuous local martingale with \( M_0 = 0 \) and \( \mathbb{E}[(\sqrt{(M_\infty)})] < \infty \), then \( M \) is a uniformly integrable martingale.

The corollary can be applied to stochastic integrals. If \( M \) is a continuous local martingale and \( H \) is a progressive process such that \( \mathbb{E}[(\int_0^t H_s^2 d\langle M \rangle_s)] < \infty \) for every \( t \geq 0 \), then \( (\int_0^t H_s dM_s)_{t \geq 0} \) is a true martingale.

**Theorem 18.4.** Assume that the filtration \( \mathcal{F} \) on \( \Omega \) is the completed canonical filtration of a standard Brownian motion \( B \). Then, any random variable \( Z \in L^2(\Omega, \mathcal{F}_\infty, \mathbb{P}) \) can be written uniquely as
\[
Z = \mathbb{E}[Z] + \int_0^\infty h_s dB_s,
\]
where \( h \in L^2(B) \).