17.1 Recap: Itô’s Formula

When \( d = 1 \), we have

\[
\frac{dF(X_t)}{dt} = F'(X_t) dX_t + \frac{1}{2} F''(X_t) d\langle X \rangle_t
\]

or in Integral form

\[
F(X_t) = F(X_0) + \int_0^t F'(X_s) dX_s + \frac{1}{2} \int_0^t F''(X_s) d\langle X \rangle_s.
\]

For \( d \geq 1 \), let \( X = (X^{(1)}, X^{(2)}, \ldots, X^{(d)}) \) be \( d \) many continuous semimartingales and \( F : \mathbb{R}^d \to \mathbb{R} \) be a twice continuously differentiable function. Then

\[
\frac{dF(X_t)}{dt} = \sum_{i=1}^d \frac{\partial F}{\partial x_i}(X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 F}{\partial x_i \partial x_j}(X_t) d\langle X^{(i)}, X^{(j)} \rangle_t
\]

\[
F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s) dX_s^{(i)} + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s) d\langle X^{(i)}, X^{(j)} \rangle_s.
\]

17.2 Lévy’s Characterization of Brownian Motion

**Theorem 17.1.** Let \( X = (X^{(1)}, X^{(2)}, \ldots, X^{(d)}) \) be \( d \) many continuous local martingales. Then the following are equivalent:

1. \( X \) is a \( (\mathcal{F}_t)_{t \geq 0} \) Brownian Motion, i.e., \( X - X_0 \) is \( d \)-dimensional Standard Brownian Motion;
2. \( \langle X^{(i)}, X^{(j)} \rangle_t = \delta_{ij} t, \forall i, j = 1, 2 \ldots d, t \geq 0 \) where \( \delta_{ij} = 1 \{i = j\} \).

**Proof.** \((2) \Rightarrow (1):\)

Recall \( \exp \{\lambda M_t - \frac{1}{2} \lambda^2 \langle M \rangle_t \} \) is a local martingale, for all \( \lambda \in i\mathbb{R} \). Take \( \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \).

Define

\[
M^\xi = \xi \cdot X_t = \sum_{i=1}^d \xi_i X_t^{(i)}, t \geq 0
\]

Then

\[
\langle M^\xi, M^\xi \rangle_t = \sum_{i,j=1}^d \xi_i \xi_j \langle X^{(i)}, X^{(j)} \rangle_t = t \|\xi\|^2
\]
In particular, \( \exp\{iM_t^x + \frac{t}{2}\|\xi\|^2\} \) is a continuous local martingale, \( \forall \xi \in \mathbb{R}^d \). Moreover, these are \( L^2 \)-martingales. For all \( 0 \leq s < t < \infty \), we have
\[
\mathbb{E}\left( \exp\left\{ iM_t^x + \frac{t}{2}\|\xi\|^2 \right\} \mid F_s \right) = \exp\left\{ iM_s^x + \frac{s}{2}\|\xi\|^2 \right\}
\]
or
\[
\mathbb{E}\left( \exp\left\{ i(M_t^x - M_s^x) \right\} \mid F_s \right) = \exp\left\{ -\frac{t-s}{2}\|\xi\|^2 \right\}
\]
Thus, \( \forall \xi \in \mathbb{R}^d, \langle \xi, X_t - X_s \rangle \perp F_s \) and follows \( N(0, (t-s)\|\xi\|^2) \). By induction, we have \( \forall 0 = t_0 < t_1 < \cdots < t_k < \infty \),
\[
\langle \xi, X_{t_{i+1}} - X_{t_i} \rangle_{i=0,1,\ldots,k-1}
\]
are independent of \( F_{t_i} \) and follows \( N(0, (t_{i+1} - t_i)\|\xi\|^2) \), \( \forall i \). Therefore, \( X_{t_{i+1}}^{(j)} - X_{t_i}^{(j)} \) are independent and follows \( N(0, t_{i+1} - t_i) \). This implies that \( X - X_0 \) is a Standard Brownian Motion. \( \blacksquare \)

### 17.3 Continuous Martingales as Time-Changed Brownian Motions

**Theorem 17.2 (Dambis–Dubins–Schwarz).** Let \( M \) be a continuous local martingale such that \( \langle M \rangle = \infty \) a.s.. Then there exists a Standard Brownian Motion \( (B_t)_{t \geq 0} \) w.r.t. some filtration s.t.
\[
M_t = \beta_{\langle M \rangle_t}, \forall t \geq 0 \quad \text{a.s.}
\]

**Proof.** Take \( r \in (0, \infty) \), define the pseudo-inverse of \( \langle M \rangle_t \) as
\[
\tau_r := \inf\{t \geq 0 \mid \langle M \rangle_t \geq r\}
\]
Then \( \tau_r \) is an a.s. finite stopping time, \( \forall r \in (0, \infty) \), \( \tau_r \) is left-continuous, and increasing in \( r \). Therefore, \( \tau_{r-} \) and \( \tau_{r+} \) exist, \( \forall r \in (0, \infty) \). Define \( \beta_r = M_{\tau_r} \), \( r \in [0, \infty) \). Note \( M^{\tau_k} \) is a \( L^2 \)-bounded martingale, \( \forall k \in (0, \infty) \). For \( 0 < q < r < k \), by Optional stopping theorem, we have
\[
\mathbb{E}(M^{\tau_k}_{r} \mid F_{\tau_q}) = M^{\tau_k}_{\tau_q} \quad \text{a.s.}
\]
Therefore
\[
\mathbb{E} (\beta_r \mid F_{\tau_q}) = \beta_q \quad \text{a.s.}
\]
Thus, \( (\beta_r)_{r \geq 0} \) is a martingale w.r.t. \( (F_r = F_{\tau_r})_{r \geq 0} \). Similarly, \( (M^2_t - \langle M \rangle_t)_{t \geq 0} \) is a continuous local martingale. Therefore,
\[
\mathbb{E}(\beta_r^2 - \langle M \rangle_{\tau_r} \mid G_q) = \beta_q^2 - \langle M \rangle_{\tau_q} \quad \text{a.s.}
\]
We claim \( \langle M \rangle_{\tau_r} = r, \forall r \in [0, \infty) \). Thus, \( \mathbb{E}(\beta_r^2 - r \mid G_q) = \beta_q^2 - q, \forall q > 0 \).

**Lemma 17.3.** For every \( a < b < \infty \),
\[
\{\langle M \rangle_b - \langle M \rangle_a = 0\} \triangleq \left\{ \sup_{t \in [a,b]} |M_t - M_a| = 0 \right\}
\]
is a null set.
From the above lemma, we have \( \beta_{r+} = M_{\tau_r} = M = \beta_r \). Further note \( \tau_r \) is left-continuous, we have \( r \mapsto \beta_r \) is continuous a.s.. Thus, \( (\beta_r)_{r \geq 0} \) is a discrete process which is a \((\mathcal{G}_r)_{r \geq 0}\) martingale with quadratic variation \( \langle \beta_r \rangle = r, \forall r \geq 0 \). By Lévy’s Characterization theorem, we have \( (\beta)_{r \geq 0} \) is a \((\mathcal{G}_r)_{r \geq 0}\) Standard Brownian Motion.

Finally, since \( \beta_r = M_{\tau_r} \), we have \( \beta_{\langle M \rangle_t} = M_t \) a.s..

\[ \text{Proof of Lemma 17.3.} \] When \( \sup_{t \in [a,b]} |M_t - M_a| = 0 \), from the approximation of \( \langle M \rangle \), \( \langle M \rangle_b - \langle M \rangle_a = 0 \) is satisfied a.s.. Now we prove the converse. Consider the continuous local martingale \( Y_t := M_{t \wedge b} - M_{t \wedge a} \). Then \( \langle Y \rangle_t = \langle M \rangle_{t \wedge b} - \langle M \rangle_{t \wedge a} = 0 \). Thererfore, \( \sup_{t \in [a,b]} |M_t - M_a| = \sup_{s \in [0,\infty)} |Y_s| = 0 \).

\[ \text{Theorem 17.4 (Burkholder, Davis, Gundy Inequalities).} \] Let \( M \) be a continuous local martingale. Then for every stopping time \( T \) and \( p > 0 \), there exists constants \( c_p \) and \( C_p \) such that

\[ c_p \| \sqrt{\langle M \rangle_T} \|_p \leq \left\| \sup_{t \leq T} |M_t| \right\|_p \leq C_p \| \sqrt{\langle M \rangle_T} \|_p. \]