12.0.1 Recap

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered probability space. We assume that the filtration is complete. Recall that, \(M = (M_t)_{t \geq 0}\) is a continuous local martingale if there exists a non-decreasing sequence of stopping times, \(T_n\), s.t. \(T_n \uparrow \infty\) and \(M^T_n\) is a uniformly integrable martingale for all \(n\).

12.0.2 Quadratic Variation Process

Theorem 12.1 (Quadratic Variation). Let \((\mathcal{F}_t)_{t \geq 0}\) be a complete filtration. If \(M\) is a continuous local martingale, there exists a unique non-decreasing adapted continuous process \(\langle M \rangle_t = \langle M, M \rangle_t\) s.t.
\[
M^2 - \langle M \rangle = (M^2 - \langle M \rangle)_{t \geq 0}
\]
is a continuous local martingale. Moreover, for any fixed \(t < \infty\) and for any sequence of partitions \(\pi_n = \{0 = t^n_0 < t^n_1 < ... < t^n_{p_n} = 1\}\) with \(\|\pi_n\| \to 0\), we have \(Q_{\pi_n}(M) = \sum_{i=1}^{p_n} (M_{t^n_i} - M_{t^n_{i-1}})^2 \to p\langle M \rangle_t\) as \(n \to \infty\).

In the last class, we showed the uniqueness of the Quadratic Variation process. Today we will prove existence. Define
\[
\tau_A = \inf\{t \geq 0 : |M_t| \geq A\} \land A.
\]
Note that \(\tau_A \uparrow \infty\) as \(A \to \infty\) and \(M^{\tau_A}\) is a bounded martingale.

First, using reduction by \(\tau_A\), we will assume that for some fixed \(T, A > 0\) we have
\[
|M_t| \leq A \text{ for all } t \in [0, T] \text{ and } M_t = M_T \text{ for all } t \geq T.
\]

Take a sequence of nested partitions,
\[
\pi_n = \{0 = t^n_0 < t^n_1 < ... < t^n_{p_n} = T\}
\]
with \(\|\pi_n\| := \sup_{1 \leq i \leq n} |t^n_i - t^n_{i-1}| \downarrow 0\) and define
\[
Q^\pi_t = \sum_{i=1}^{p_n} (M_{t^{\pi}_i} - M_{t^{\pi}_{i-1}})^2, \quad t \in [0, T].
\]

We want to show that \(Q^\pi_t\) converges to a continuous path as \(n \to \infty\). It is enough to show that
\[
\mathbb{E}(\sup_{m \geq n} \|Q^\pi_t - Q^\pi_{m,t}\|_{L^2}) \to 0 \text{ in } L^2 \text{ as } n \to \infty.
\]
To show this, it would help to get rid of the supremum and employ theorems about martingales. Doob’s $L^2$-maximal inequality tells us that

\[
\| \sup_{0 \leq t \leq T} |M_t| \|^2 \leq 4 \| M_T \|^2.
\]

We also exploit a simple trick with polynomials:

\[(x - y)^2 = x^2 - y^2 - 2xy + 2y^2 = x^2 - y^2 - 2y(x - y) .\]

Using this, we have that on the LHS, the quadratic variation $Q_t^{\pi_n}$ can be written as

\[
Q_t^{\pi_n} = M_t^2 - 2 \sum_{i=1}^{n} M_{t_{i-1}} (M_{t_{i}} - M_{t_{i-1}}) = M_t^2 - 2X_t^{\pi_n}
\]

where

\[
X_t^{\pi_n} := \sum_{i=1}^{n} M_{t_{i-1}} (M_{t_{i}} - M_{t_{i-1}}), \quad t \geq 0
\]

is a martingale. Rearranging terms, we have that $M_t^2 - Q_t^{\pi_n} = 2X_t^{\pi_n}$. Hence, we can apply Doob, yielding for $m > n$

\[
\mathbb{E}( \sup_{0 \leq t \leq T} |X_t^{\pi_n} - X^{\pi_m}|^2 ) \leq 4 \mathbb{E}(X_T^{\pi_n} - X_T^{\pi_m})^2 = 4 \mathbb{E} \left( \sum_{i=1}^{m} (M_{t_{i}} - M_{t_{i-1}})(M_{t_{i}} - M_{t_{i-1}}) \right)^2
\]

where $t_{i-1}^* := \sup \{ t \in M_n \mid t \leq t_{i-1}^{m} \}$. Conveniently, the expectation of the crossproducts is zero, so we only need to show

\[
\mathbb{E}(X_T^{\pi_n} - X_T^{\pi_m})^2 = \mathbb{E} \sum_{i=1}^{m} (M_{t_{i}} - M_{t_{i-1}})^2(M_{t_{i}} - M_{t_{i-1}})^2
\]

is bounded. Now,

\[
\mathbb{E}(X_T^{\pi_n} - X_T^{\pi_m})^2 \leq \mathbb{E} \left( \sup_{i} (M_{t_{i}} - M_{t_{i-1}})^2 \cdot \sum_{i=1}^{m} (M_{t_{i}} - M_{t_{i-1}})^2 \right)
\]

\[
\leq \sqrt{ \mathbb{E} \left( \sup_{i} (M_{t_{i}} - M_{t_{i-1}})^4 \right) \mathbb{E} \left( \sum_{i=1}^{m} (M_{t_{i}} - M_{t_{i-1}})^2 \right)^2 },
\]

where we used Cauchy-Schwarz in the last inequality.

Here the first term inside the radical goes to 0, by uniform continuity of the sample path and DCT. Moreover, we have

\[
\mathbb{E} \left( \sum_{i=1}^{m} (M_{t_{i}} - M_{t_{i-1}})^2 \right)^2 = \sum_{i=1}^{m} \mathbb{E} \left( (M_{t_{i}} - M_{t_{i-1}})^2 \mathbb{E} \left( \sum_{j \geq i} (M_{t_{j}} - M_{t_{j-1}})^2 \mid \mathcal{F}_{t_{i}} \right) \right)
\]

\[
= \sum_{i=1}^{m} \mathbb{E} \left( (M_{t_{i}} - M_{t_{i-1}})^2(M_T - M_{t_{i-1}})^2 \right),
\]

\[= \sum_{i=1}^{m} \mathbb{E} \left( (M_{t_{i}} - M_{t_{i-1}})^2 \right) \mathbb{E} \left( (M_T - M_{t_{i-1}})^2 \right),
\]
which is bounded above by

\[
4A^2 \mathbb{E} \left( \sum_{i=1}^{m} (M_{t_i}^m - M_{t_{i-1}}^m)^2 \right) = 4A^2 \mathbb{E}(M_T^2) \leq 4A^4.
\]

The boundedness of the quantity above implies that \(\sup_{m \geq n} \|Q_{\pi}^n - Q_{\pi}^m\|_\infty \|_2 \to 0\) as \(n \to \infty\). By choosing an appropriate subsequence \(n_k \uparrow \infty\), we have that

\[
\sum_{k=1}^{\infty} \|Q_{\pi_{nk}}^n - Q_{\pi_{nk+1}}^n\|_\infty \|_2 \leq 1,
\]

implying that \(\sum_{k=1}^{\infty} \sup_{0 \leq t \leq T} |Q_{\pi_{nk+1}}^n - Q_{\pi_{nk+1}}^n| < \infty\) a.s. Thus, there exists a continuous function \(A\), s.t.

\[
\|Q_{\pi_{nk}}^n - A\|_\infty \to 0 \text{ in } \mathcal{L}^2.
\]

It is easy to see that \((A_t)_{0 \leq t \leq T}\) is continuous and adapted. Also, \((M_t^2 - Q_{\pi_{nk}}^n)_{t \in [0,T]}\) is a martingale for all \(k\), i.e.,

\[
\mathbb{E}(M_t^2 - Q_{\pi_{nk}}^n | \mathcal{F}_s) = M_s^2 - Q_{\pi_{nk}}^n
\]

Taking limit, we get that \((M_t^2 - A_t)_{0 \leq t \leq T}\) is a martingale.

**Corollary 12.2.** If \(M\) is a continuous local martingale with \(M_0 = 0\) and \(\langle M \rangle \equiv 0\), then \(M_t = 0\) for all \(t \geq 0\).

**Proof.** We prove the forward direction only, since the reverse direction is obvious. Suppose \(\langle M \rangle \equiv 0\), implying that \((M_t^2)\) is a continuous nonnegative local martingale as well as a supermartingale (by a theorem proved in a previous class). Since \(M_t^2\) is a nonnegative supermartingale, we can exploit the central property of supermartingales: \(\mathbb{E}(M_t^2) \leq \mathbb{E}(M_0^2) = 0\), where the equality holds by assumption. Thus, for all \(t\), \(M_t = 0\), as desired. \(\blacksquare\)

**Definition 12.3.** For two local martingales \(M, N\), we define

\[
\langle M, N \rangle_t := \frac{1}{2} (\langle M + N \rangle_t - \langle M \rangle_t - \langle N \rangle_t)
\]

to be the covariation process.

The following result gives a natural condition under which a local martingale becomes a true martingale.

**Theorem 12.4 (\(L^2\) bounded martingales).** Let \(M\) be a continuous local martingale with \(M_0 \in \mathcal{L}^2\). The following are equivalent.

1. \(M\) is an \(\mathcal{L}^2\)-bounded martingale.
2. \(\mathbb{E}(\langle M \rangle_\infty) < \infty\).

Similarly, the following are equivalent.

1. \(M\) is a martingale with \(\mathbb{E}[M_t^2] < \infty\) for all \(t\).
2. \(\mathbb{E}(\langle M \rangle_t) < \infty\) for all \(t\).