11.1 Recap

Last time, we have talked about FV process, an adaptive process whose trajectories have finite variation almost surely. And we have shown that, if \( A_t \) is a FV process, and \( H_t \) is progressively measurable, then \((H \cdot A)_t\) is a FV process, as long as

\[
\int_0^t |H_s(\omega)| |dA_s(\omega)| < \infty, \quad \forall \, t \geq 0, \omega \in \Omega.
\]

11.2 Local martingale

For any adapted process \( X = (X_t)_{t \geq 0} \) and a stopping time \( T \), let \( X_T := (X_{t \wedge T})_{t \geq 0} \) be the stopped process at time \( T \). We shall notice that

\[
(X^S_T) = X^{S \wedge T} = (X^T)^S
\]

for any two stopping time \( T \) and \( S \).

**Definition 11.1 (continuous local martingale).** An adapted process \( M = (M_t)_{t \geq 0} \) is a continuous local martingale starting at 0, if \( M_0 = 0 \), and there is a non-decreasing sequence of stopping time \( T_n \uparrow \infty \) a.s., such that \( \{M_{T_n} : n \geq 1\} \) are u.i. martingales. In this case, we say \((T_n)_{n \geq 0}\) reduces \( M \).

**Remark 11.2.** Here are some properties of local martingales.

1. Any continuous martingale \( M \) is a continuous local martingale. For instance, we can take \( T_n = \inf \{t : |M_t| \geq n\} \). Then \( M_{T_n} \) is a bounded martingale, and thus u.i.

2. A continuous local martingale may not be a martingale. A counterexample is \( M_t = ZB_t \), with \( \mathbb{E}|Z| = \infty \) and \( Z \perp \sigma(B_t, t \geq 0) \). The key point is that, even though \( M_t \) has martingale structure, the first-order moment is not finite, so it is not a martingale. But \( M_{T_n} \) is a bounded martingale.

3. If \((T_n)_{n \geq 0}\) reduces \( M \), and \( S_n \uparrow \infty \) a.s. is a sequence of stopping time, then \((T_n \wedge S_n)_{n \geq 0}\) also reduces \( M \).

4. All continuous local martingale (starting from 0) form a vector space.

**Proposition 11.3.** Let \( M \) be a continuous local martingale.

(i) If \( M_t \geq 0 \) a.s. and \( \mathbb{E}|M_t| < \infty \), then \( M \) is a super-martingale.
(ii) A dominated local martingale is a martingale. More precisely, if $|M_t| \leq Y$ for all $t \geq 0$ and $\mathbb{E}Y < \infty$, then $M$ is a continuous martingale.

(iii) The sequence of stopping time $\tau_n := \inf\{t \geq 0 : |M_t| \geq n\}$ reduces $M$.

Proof. (i) Let $T_n$ be a stopping time reduce $M$. Since $M^{T_n}$ is a u.i. martingale, $\mathbb{E}[M_{t \wedge T_n} | \mathcal{F}_s] = M_s$ for all $s \leq t$. Notice that $M_t \geq 0$ and $T_n \uparrow \infty$. By Fatou’s lemma

$$\mathbb{E}(M_t | \mathcal{F}_s) = \mathbb{E}\left(\lim_{n \to \infty} M_{t \wedge T_n} | \mathcal{F}_s\right) \leq \lim_{n \to \infty} \mathbb{E}(M_{t \wedge T_n} | \mathcal{F}_s) = \lim_{n \to \infty} M_{s \wedge T_n} = M_s.$$ 

(ii) Just use DCT.

(iii) It is easy to see that $M^{\tau_n}$ is a martingale, and $|M_t^{\tau_n}| \leq n$ is bounded. So $M^{\tau_n}$ is a u.i. martingale. Also, $\tau_n \uparrow \infty$. By definition, $(\tau_n)_{n \geq 1}$ reduces $M$. □

**Theorem 11.4.** If $M$ is a FV continuous local martingale starting from 0, then $M_t = 0$ a.s.

Proof. Since $M$ has finite variation, $\int_0^t |dM_s|$ makes sense for all $t \geq 0$. Define $\tau_n := \inf\{t \geq 0 : \int_0^t |dM_s| \geq n\}$. Then $\tau_n \uparrow \infty$ are stopping times. Notice that

$$|M_t| = \left| \int_0^t dM_s \right| \leq \int_0^t |dM_s|.$$ 

So, $\tau_n$ reduces $M$. For a fixed $k \geq 1$, let $N_t := M_{t \wedge \tau_k}$ be a bounded martingale. Then for any partition $0 = t_0 < t_1 < \cdots < t_n = t$, we have

$$\mathbb{E}(N_{t_i} - N_{t_{i-1}})^2 = \mathbb{E}\left[\mathbb{E}\left[N_{t_i}^2 - 2N_{t_i}N_{t_{i-1}} + N_{t_{i-1}}^2 | \mathcal{F}_{t_{i-1}}\right]\right]$$

$$= \mathbb{E}\left[N_{t_{i-1}}^2 - 2N_{t_{i-1}}\mathbb{E}[N_{t_i} | \mathcal{F}_{t_{i-1}}] + \mathbb{E}[N_{t_i}^2 | \mathcal{F}_{t_{i-1}}]\right]$$

$$= \mathbb{E}N_{t_{i-1}}^2 - 2\mathbb{E}N_{t_{i-1}}^2 + \mathbb{E}N_{t_i}^2$$

$$= \mathbb{E}N_{t_i}^2 - \mathbb{E}N_{t_{i-1}}^2.$$ 

Therefore,

$$\mathbb{E}N_t^2 = \sum_{i=1}^n \mathbb{E}(N_{t_i} - N_{t_{i-1}})^2$$

$$\leq \mathbb{E}\left(\max_{1 \leq i \leq n} |N_{t_i} - N_{t_{i-1}}| \cdot \sum_{j=1}^n |N_{t_j} - N_{t_{j-1}}|\right)$$

$$\leq k\mathbb{E}\max_{1 \leq i \leq n} |N_{t_i} - N_{t_{i-1}}|$$

$$\to 0$$

by DCT and the fact that $N_t$ is continuous a.s. □

**Theorem 11.5** (quadratic variation). For any continuous local martingale $M$, there is a unique continuous non-decreasing adapted process $\langle M, M \rangle_t =: \langle M \rangle_t$, with $\langle M \rangle_0 = 0$ a.s., such that $(M_t^2 - $
\langle M \rangle_t \quad t \geq 0 \text{ is a continuous local martingale. Moreover, if } \pi_n = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{p_n}^{(n)} = t\} \text{ is a sequence of increasing partition with } ||\pi_n|| \downarrow 0, \text{ then}

\sum_{i=1}^{p_n} \left(M_{t_i^{(n)}} - M_{t_{i-1}^{(n)}}\right)^2 \overset{p_n}{\to} \langle M \rangle_t.

\textit{Proof. Uniqueness.} Let } A \text{ and } A' \text{ be two continuous, adapted, non-decreasing process, such that } A_0 = A'_0 = 0, \text{ and } (M_t^2 - A_t)_{t \geq 0} \text{ and } (M_t^2 - A'_t)_{t \geq 0} \text{ are continuous local martingales. Then}

A_t - A'_t = (M_t^2 - A'_t) - (M_t^2 - A_t)

\text{is a continuous local martingale, with finite variation, since } A \text{ and } A' \text{ are non-decreasing. Notice that } A_0 - A'_0 = 0. \text{ By the theorem above, we know } A_t - A'_t = 0 \text{ a.s. for all } t \geq 0. \quad \blacksquare