8.1 Recap

Recall the following Lemma:

**Lemma 8.1.** Let \((X_t, \mathcal{F}_t)_{t \geq 0}\) be a supermartingale and \(D\) a countable dense subset of \([0, \infty)\). If \(f: D \to \mathbb{R}\) satisfies:

1. \(\sup_{s \in D \cap [0,T]} |f(x)| < \infty, \forall \ T < \infty\)
2. The upcrossing number \(U_{ab} (D \cap [0, T]) < \infty, \forall \ T \in D, a, b \in \mathbb{Q}\),

then \(f(t^+) := \lim_{s \uparrow t} X_s\) exists for \(t \geq 0\) and \(f(t^-) := \lim_{s \downarrow t} X_s\) exists for \(t > 0\).

8.2 Martingale Convergence Theorems

We begin with the following result, which we will use to construct RCLL modifications to continuous time martingales.

**Theorem 8.2.** Let \((X_t, \mathcal{F}_t)_{t \geq 0}\) be a supermartingale and \(D\) a countable dense subset of \([0, \infty)\). Then:

1. Almost surely, \((X_t)_{t \in D}\) satisfies \(X_{t^+} := \lim_{s \in D} X_s\) exists for \(t \geq 0\) and \(X_{t^-} := \lim_{s \in D} X_s\) exists for \(t > 0\).

2. Almost surely, \(X_t \geq \mathbb{E}(X_{t^+} | \mathcal{F}_t)\) for \(t \geq 0\). Here, equality holds if the map \(t \mapsto \mathbb{E}(X_t)\) is right continuous. The process \((X_{t^+}, \mathcal{F}_{t^+})\) is also a supermartingale.

Before we prove this theorem, note that the map \(t \mapsto \mathbb{E}(X_t)\) is always right continuous if \((X_t)\) is a martingale.

**Proof.** We define the set

\[
N := \bigcup_{T \in D} \{ \sup_{s \in D \cap [0,T]} |X_s| = \infty \} \cup \bigcup_{a,b \in \mathbb{Q}} \{ U_{ab} (D \cap [0, T]) = \infty \}.
\]

To prove statement the first statement, it suffices to show that \(N\) is a null set. However, since \(D\) is a countable set, we have the following discrete-time martingale results:

- \(\mathbb{P}(\sup_{D \cap [0,T]} |X_s| > \lambda) \leq \frac{1}{\lambda} (\mathbb{E}X_0 + 2 \mathbb{E} |X_T|)\)
• $\mathbb{E}(U_{ab}^X(\mathcal{D} \cap [0, T])) \leq \frac{\mathbb{E}(\lvert X_T - a \rvert)}{b-a} \leq \frac{\mathbb{E}(\lvert X_T \rvert) + \mathbb{E}(\lvert a \rvert)}{b-a}$.

The result follows by combining these facts with the previous Lemma.

We now show the second part of the Theorem. It suffices to show that $X_{t_n} \xrightarrow{L^1} X_t$ when $t_n \downarrow t$ for any sequence $(t_n)_n \in \mathcal{D}$. Recall the supermartingale property for $(X_t, \mathcal{F}_t)_t$: if $s < t$, we have $X_t \geq \mathbb{E}(X_s \mid \mathcal{F}_t)$.

Denote by $Y_{-k} := X_{t_k}$ and $\mathcal{H}_{-k} := F_{t_k}$. The process $(Y_{-k}, \mathcal{H}_{-k})_k$ is a discrete time reverse supermartingale. Thus, $(Y_{-k}, \mathcal{H}_{-k})_k$ converges both a.s. and in $L^1$. Obviously, $Y_\infty = X_{t^+}$, which establishes the existence of the claimed limit.

Now, consider $W = X_t - \mathbb{E}(X_{t^+} \mid \mathcal{F}_t) \geq 0$. If $t \mapsto \mathbb{E}(X_t)$ is right continuous, we have that $\mathbb{E}(W) = \mathbb{E}(X_t) - \mathbb{E}(X_{t^+}) = 0$; hence $W = 0$ a.s.

Finally, take $A \in \mathcal{F}_{s^+} \subseteq \mathcal{F}_{t^+}$, with $s < t$. If $s_n \downarrow s$ and $t_n \downarrow t$ with $s_n \leq t_n$ for all $n$, and $(s_n, t_n)_n \in \mathcal{D} \times \mathcal{D}$, it is obvious that $\mathbb{E}(X_{s^+} \mid A) \geq \mathbb{E}(X_{t^+} \mid A)$, and the result follows.

Note that if the original filtration is right continuous, this theorem implies that a.s., $X_t = X_{t^+}$ for all $t \geq 0$. From now on, we will make the assumption that filtrations are right continuous.

The next theorem follows from the previous, and we omit the proof.

**Theorem 8.3.** Let $(X_t, \mathcal{F}_t)_{t \geq 0}$ be a supermartingale, where $\{\mathcal{F}_t\}_t$ is right continuous and complete; and suppose that the map $t \mapsto \mathbb{E}(X_t)$ is right continuous. Then $(X_t)_t$ has an RCLL modification which is also an $\{\mathcal{F}_t\}_t$ supermartingale.

We now prove continuous analogs of some discrete time martingale convergence results. For these results, we use the analytic technique of using the convergence on a countable dense subset and the right continuity assumption (recall that we can make this assumption “for free” as per the previous results) to establish a.e. convergence. This is a standard technique and in these notes we omit the details.

**Theorem 8.4.** Let $(X_t, \mathcal{F}_t)_{t \geq 0}$ be a supermartingale with $\sup_{\mathbb{R}^+} \mathbb{E}(\lvert X_t \rvert) < \infty$ and with almost right continuous sample paths. Then $X_t \xrightarrow{a.s.} X_\infty$, with $X_\infty \in L^1$.

**Proof.** Let $\mathcal{D}$ be a countable dense subset of $[0, \infty)$. We have that $\mathbb{E}(U_{ab}^X(\mathcal{D} \cap [0, T])) \leq \frac{\mathbb{E}(\lvert X_T \rvert + \mathbb{E}(\lvert a \rvert))}{b-a}$ for all $T$. From the assumption that $\sup_{\mathbb{R}^+} \mathbb{E}(\lvert X_t \rvert) < \infty$, we can upper bound this by $\frac{c+\lvert a \rvert}{b-a}$ for some universal constant $c$. Thus, as $T \to \infty$, we get that $U_{ab}(D) \leq \frac{c+\lvert a \rvert}{b-a}$ for all $a, b \in \mathbb{R}$; and thus also for all $a, b \in \mathbb{Q}$.

Since sample paths are right continuous, standard analysis arguments imply that $X_t \xrightarrow{a.s.} X_\infty$ with $X_\infty \in L^1$ as $t \to \infty$.

The following theorem has essentially the same proof, but instead uses Doob’s $L^p$ maximal inequality. We omit the proof for brevity.

**Theorem 8.5.** If $\sup_{t \in \mathbb{R}^+} \mathbb{E}(\lvert X_t \rvert^p) < \infty$ with $1 < p < \infty$ and $(X_t)_t$ is a supermartingale, then $X_t \xrightarrow{L^p, a.s.} X_\infty$ with $X_\infty \in L^1$. 

8.3 Uniform Integrability

8.3.1 Recap

Let $\Lambda$ be a (potentially uncountable) indexing set.

**Definition 8.6.** The random variables $(X_\lambda)_{\lambda \in \Lambda}$ are **tight** if
\[
\lim_{x \uparrow \infty} \sup_{\lambda \in \Lambda} \mathbb{P}(|X_\lambda| > x) = \mathbb{E}(1_{|X_\lambda| > x}) = 0.
\]

This can also be thought of as "$L^0$ control."

**Definition 8.7.** The random variables $(X_\lambda)_{\lambda \in \Lambda}$ are **uniformly integrable (UI)** if
\[
\lim_{x \uparrow \infty} \sup_{\lambda \in \Lambda} \mathbb{E}(1_{|X_\lambda| > x} | X_\lambda) = 0.
\]

This can also be thought of as "$L^1$ control."

Obviously, UI implies tightness.

We briefly gives some conditions and results on UI.

1. $|X_\lambda| \leq Y \ \forall \lambda, \ Y \in L^1 \Rightarrow$ UI.
2. $X_n \overset{L^1}{\to} X_\infty \Rightarrow$ UI.
3. $X_n \overset{p}{\to} X_\infty$ and UI $\Rightarrow X_n \overset{L^1}{\to} X_\infty$
4. $(\mathbb{E}(X | \mathcal{F}_\lambda))_{\lambda \in \Lambda}$ are UI.

8.3.2 UI and Continuous Time Martingales

**Definition 8.8.** A martingale $(M_t, \mathcal{F}_t)_{t \geq 0}$ is **closed** if $\exists M_\infty \in L^1$ with $M_t = \mathbb{E}(M_\infty | \mathcal{F}_t)$ a.s. for all $t \geq 0$.

As in discrete time, we need UI to discuss $L^1$ convergence of martingales:

**Theorem 8.9.** Let $(M_t, \mathcal{F}_t)_{t \geq 0}$ be a martingale with almost sure right continuous sample paths. Then the following are equivalent:

1. $(M_t)_t$ is closed.
2. $(M_t)_t$ is UI.
3. $M_t \overset{L^1}{\to} M_\infty$.

**Proof.** (1) $\Rightarrow$ (2): Take $M_t = \mathbb{E}(M_\infty | \mathcal{F}_t)$ for all $t \geq 0$ and use condition 4 above.

(2) $\Rightarrow$ (3): UI implies $\sup_{t \geq 0} \mathbb{E}(|M_t|) < \infty \Rightarrow M_t \overset{a.s.}{\to} M_\infty$. Now use condition 3 above.

(3) $\Rightarrow$ (1): Let $M_t = \mathbb{E}(M_{t+k} | \mathcal{F}_t)$ for all $k \geq 0$ and let $k \uparrow \infty$, so that $M_{t+k} \overset{L^1}{\to} M_\infty$. 

$\blacksquare$
8.4 Optional Stopping

**Theorem 8.10** (Optional Stopping Theorem). If \((M_t, \mathcal{F}_t)_{t \geq 0}\) is a UI martingale with a.s. right continuous sample paths and \(0 \leq S < T \leq \infty\) are stopping times, then \(M_S = \mathbb{E}(M_T \mid \mathcal{F}_S)\).

**Proof idea:** We use discrete time martingales indexed over \(\mathbb{Q}\), and sequences of stopping times \(S_n \to S\) and \(T_n \to T\) with \(S_n \leq T_n\) for all \(n\). And then extend to \(\mathbb{R}\) since \(\mathbb{Q}\) is dense. ■

Note that if \(T = \infty\), we set \(M_T = M_\infty\).

We have the following corollary:

**Corollary 8.11.** Fix \(a \in [0, \infty)\). If \((M_t, \mathcal{F}_t)_{t \geq 0}\) is a martingale, then \((M_{t \wedge a}, \mathcal{F}_t)_{t \geq 0}\) is a UI martingale.

Thus, if \(S \leq T < \infty\) a.s., we have \(M_S = \mathbb{E}(M_T \mid \mathcal{F}_S)\) a.s..

For the rest of this lecture notes, we will use the fact (to be proven in another lecture in the future) that if \((B_t)_{t \geq 0}\) is a Brownian motion w.r.t. \((\mathcal{F}_t)_{t \geq 0}\), then \((B_{t \wedge T}, \mathcal{F}_t)_{t \geq 0}\) is a martingale if \(T\) is a stopping time.

Let’s consider an example.

Let \((B_t, \mathcal{F}_t)_{t \geq 0}\) be SBM. Let \(H_a = \inf\{t \geq 0 : B_t = a\}\). Fix \(a < 0 < b\) and set \(T := H_a \wedge H_b\).

Obviously, \(\mathbb{E}(B_T) = \mathbb{E}(B_0) = 0\), so that

\[
\mathbb{E}(B_T) = a \mathbb{P}(H_A < H_b) + b \mathbb{P}(H_b < H_A).
\]

Thus, we have that \(\mathbb{P}(H_A < H_b) = \frac{b}{b-a}\) and \(\mathbb{P}(H_b < H_A) = \frac{a}{b-a}\).

Now, set \(U_a = H_a \wedge H_{-a}\), for \(a > 0\). Let’s use the exponential martingale \(M_t = \exp(\lambda B_t - \lambda^2 t/2)\).

Note that this is a martingale on the same filtration and it is UI.

So we have that \(\mathbb{E}(\exp(\lambda B_{U_a} - \lambda^2 U_a/2)) = 1\). By symmetry, \(\mathbb{P}(H_a < H_{-a}) = \mathbb{P}(H_{-a} < H_a) = \frac{1}{2}\).

Thus, we can evaluate: \(\mathbb{E}(\exp(-\lambda^2 U_a/2))(\frac{1}{2}e^{\lambda a} + \frac{1}{2}e^{-\lambda a}) = 1\); hence we get the generating function \(\mathbb{E}e^{-\theta U_a} = 1/\cosh(a\sqrt{2\theta})\), for \(\theta > 0\).