7.0.1 Martingales

Let \((X_t^{(n)})_{0 \leq t \leq T}\) be \(\{F_t\}\)-adapted. If \((X_t)_{t \geq 0}\) has right continuous paths almost surely, then

\[X_t^{(n)} \xrightarrow{a.s.} X_t, \forall t \in [0, T]\]

**Definition 7.1.** Let \(\{F_t\}_{t \geq 0}\) be a filtration. The stochastic process \((X_t)_{t \geq 0}\) is a \((F_{t \geq 0})_{t \geq 0}\) martingale if:

- (adapted) \((X_t)_{t \geq 0}\) is \(\{F_{t \geq 0}\}_{t \geq 0}\)-adapted.
- (integrable) \(X_t \in L^1, \forall t \geq 0\), i.e. \(E|X_t| < \infty, \forall t\)
- (conditional mean) \(E(X_t | F_s) = X_s\) a.s., \(\forall 0 \leq s < t < \infty\)

The same holds for submartingales and supermartingales, respectively, if

- \(E(X_t | F_s) \geq X_s\) a.s., \(\forall 0 \leq s < t < \infty\)
- \(E(X_t | F_s) \leq X_s\) a.s., \(\forall 0 \leq s < t < \infty\)

**Definition 7.2.** A process \((Z_t)_{t \geq 0}\) has independent increments w.r.t. \(\{F_t\}_{t \geq 0}\) if it is adapted and

\[Z_t - Z_s \perp F_s, \forall 0 \leq s < t < \infty\]

**Example 7.3.** Take any \(f \in L^2[0, T]\). Let

\[Z_t = G(f \mathbb{1}_{[0,t]}) = \int_0^t f_s dB_s \sim N(0, \int_0^t f_s^2 du), \forall t \in [0, T]\]

where \(G\) is Gaussian white noise.

- \((Z_t)_{t \geq 0}\) has independent increments with respect to \(\{F_t\}_{t \geq 0}\) where \(F_t = \sigma(B_s, s \leq t), t \geq 0\).
  Hint: \(Z_t - Z_s \sim N(0, \int_s^t f_u^2 du), s \leq t\) and \(\perp Z_{s_1}, ..., Z_{s_k}\) for \(s_1, ..., s_k \leq s\).

**Proposition 7.4.** Let \((Z_t)_{t \geq 0}\) have independent increments with respect to \(\{F_t\}_{t \geq 0}\)

i) If \(E|Z_t| < \infty, \forall t \geq 0\), then \((Z_t - EZ_t)_{t \geq 0}\) is a \((F_{t \geq 0})_{t \geq 0}\) martingale.

ii) If \(E|Z_t|^2 < \infty, \forall t \geq 0\), and \(EZ_t = 0, \forall t \geq 0\), then \((Z_t^2 - EZ_t^2)_{t \geq 0}\) is a \((F_{t \geq 0})_{t \geq 0}\) martingale.

iii) If \(E(e^{\lambda Z_t}) < \infty, \forall t \geq 0\), then \((e^{\lambda Z_t} - \frac{e^{\lambda Z_t}}{Ee^{\lambda Z_t}})_{t \geq 0}\) is a \((F_{t \geq 0})_{t \geq 0}\) martingale.

**Corollary 7.5.** For \(F_t = \sigma(B_s, s \leq t), \forall t \geq 0\), and \((B_t)_{t \geq 0}\) is a SBM, then
\((B_t)_{t \geq 0}, (B_{t^2-t})_{t \geq 0}, (e^{\lambda B_t} - \frac{e^{\lambda B_t}}{Ee^{\lambda B_t}})_{t \geq 0}\) are all \((F_{t \geq 0})_{t \geq 0}\) martingales.

**Proof.** We will prove ii), but i) and iii) are similar.

Let \(M_t = Z_t^2 - EZ_t^2\)
and
\[ M_t - M_s = Z_t^2 - Z_s^2 - \mathbb{E}(Z_t^2 - Z_s^2) = (Z_t - Z_s)(Z_t - Z_s + 2Z_s) - \mathbb{E}((Z_t - Z_s)(Z_t - Z_s + 2Z_s)) = (Z_t - Z_s)^2 + 2Z_s(Z_t - Z_s) - \mathbb{E}((Z_t - Z_s)^2 + 2Z_s(Z_t - Z_s)) \]

Fix \( 0 \leq s < t \).
\[ \mathbb{E}(M_t - M_s \mid \mathcal{F}_s) = \mathbb{E}(Z_t - Z_s)^2 + 2Z_s \mathbb{E}(Z_t - Z_s) - \mathbb{E}(Z_t - Z_s)^2 + \mathbb{E}(2Z_s(Z_t - Z_s)) = 0 \text{ a.s.} \]

\[ \square \]

**Definition 7.6.** A stochastic process \((X_t)_{t \geq 0}\) is a \((\mathcal{F}_t)_{t \geq 0}\)-BM if

- \((X_T - X_0)_{t \geq 0}\) is a BM
- \((X_t)_{t \geq 0}\) is \((\mathcal{F}_t)_{t \geq 0}\) adapted
- \(X_t - X_s \perp \mathcal{F}_s, \forall s \geq 0\), i.e. \((X_t)_{t \geq 0}\) has independent increments with respect to \((\mathcal{F}_t)_{t \geq 0}\).

**Theorem 7.7 (Doob’s Maximal Inequality).** If \((X_n)_{n \geq 0}\) is a \((\mathcal{F}_n)_{n \geq 0}\) submartingale, then
\[ \lambda \mathbb{P}(\sup_{0 \leq i \leq n} |X_i| > \lambda) \leq 2\mathbb{E}|X_n| + \mathbb{E}|X_0|, \forall n \geq 0 \]

**Theorem 7.8 (Doob’s L_p Maximal Inequality).** If \((X_n, \mathcal{F}_n)_{n \geq 0}\) is a non-negative submartingale, then
\[ \| \sup_{0 \leq i \leq n} |X_i| \|_p \leq \frac{p}{p-1} \|X_n\|_p \text{ where } \|Y\|_p = (\mathbb{E}|Y|^p)^{1/p}, \forall n \geq 0, p > 1. \]

**Lemma 7.9.** If \((X_t, \mathcal{F}_t)_{t \geq 0}\) is a martingale and \(\phi\) is a convex function, then \((\phi(X_t), \mathcal{F}_t)_{t \geq 0}\) is a submartingale and
\[ \mathbb{E}(\phi(X_t) \mid \mathcal{F}_s) \geq \phi(\mathbb{E}(X_t \mid \mathcal{F}_s)) = \phi(X_s) \text{ a.s.} \]

by Conditional Jensen’s Inequality.

**Lemma 7.10.** If \((X_t, \mathcal{F}_t)_{t \geq 0}\) is a submartingale, \(\phi\) is convex and non-decreasing, then \((\phi(X_t), \mathcal{F}_t)_{t \geq 0}\) is a submartingale.

**Corollary 7.11.** The following two implications hold:

1. if \((X_t, \mathcal{F}_t)_{t \geq 0}\) martingale \(\Rightarrow ((X_t, \mathcal{F}_t)_{t \geq 0}\) submartingale
2. if \((X_t, \mathcal{F}_t)_{t \geq 0}\) submartingale \(\Rightarrow (X^+_t, \mathcal{F}_t)_{t \geq 0}\) submartingale.

Fix \( T < \infty \) and let \( D \subseteq [0, T] \) be countable. Find finite \( D_n \) and let \( D_n \uparrow D \) as \( n \uparrow \infty \) and \( T \in D_n, \forall n \in \mathbb{N} \). Then we can replace the index of Doob’s maximal inequality and Doob’s L^p Maximal Inequality by \( i \in D_n \cap [0, T] \) and as \( n \uparrow \infty \), this goes to \( i \in D \cap [0, T] \).

**Theorem 7.12.** i) If \((X_t, \mathcal{F}_t)_{t \geq 0}\) is a submartingale with right continuous paths, then
\[ \lambda \mathbb{P}(\sup_{0 \leq s \leq t} |X_s| > \lambda) \leq 2\mathbb{E}|X_t| + \mathbb{E}|X_0|, \forall t \geq 0 \]

ii) If \((X_t, \mathcal{F}_t)_{t \geq 0}\) is a non-negative submartingale with right continuous paths, then
\[ \| \sup_{0 \leq s \leq t} |X_s| \|_p \leq \frac{p}{p-1} \|X_t\|_p, \forall p > 1, t \geq 0 \]

therefore we can bound by the last point.
7.0.2 Upcrossings

Recall that an upcrossing is the number of times going below $a$ and above $b$, and recall that the Upcrossing Lemma from MATH 561 states that

$$(b - a)\mathbb{E}(U_{T}[a,b]) \leq \mathbb{E}|X_t - a|, \forall a < b$$

**Theorem 7.13 (Doob’s Upcrossing Inequality).** Let $D$ be a countable set and $U_{ab}(D \cap [0,T])$ is the number of upcrossings from $a$ to $b$ for the process $(X_s)_{s \in D \cap [0,t]}$, then

$$(b - a)\mathbb{E}U_{ab}(D \cap [0,T]) \leq \mathbb{E}|X_T - a|$$

**Proposition 7.14.** If $f$ is a function on a countable dense subset $D$ of $[0,T]$ such that
- $\sup_{s \in D} |f_s| < \infty$
- $U_{ab}^f(D) < \infty, \forall a < b, a, b, \in \mathbb{Q}$

Then $f(t^+) = \lim_{s \downarrow t, s \in D} f(s)$ is well-defined \(\forall t \geq 0\) and $f(t^-) = \lim_{s \uparrow t, s \in D} f(s)$ is well-defined \(\forall t \geq 0\).

Moreover, $g(t) = f(t^+), t \in [0,T]$, is a RCLL (right continuous with left limits) function and $g(t) = f(t), \forall t \in D$. 