6.0.1 Filtrations

**Theorem 6.1 (Doob’s $L^p$-inequality).** If $(M_n, F_n)_{n \geq 0}$ is a martingale, then

$$|| \sup_{0 \leq k \leq n} |M_k||_p \leq \frac{p}{p-1} ||M_n||_p$$

where $||X||_p = (\mathbb{E}|X|^p)^{1/p}$.

Therefore, we just need to control one point.

**Definition 6.2.** A filtration is an increasing sequence of σ-fields $\{F_t\}_{t \geq 0}$ such that

$$F_s \subseteq F_t, \forall s, t \in [0, \infty)$$

Recall that $F_t = \bigcap_{\epsilon > 0} F_{t+\epsilon}$.

**Definition 6.3.** For a stochastic process $(X_t)_{t \geq 0}$, $F_t = \sigma(X_s, s \leq t), t \geq 0$ is called the canonical filtration.

**Definition 6.4.** A stochastic process $(X_t)_{t \geq 0}$ is measurable w.r.t. $F \otimes B$ if

$$(\omega, t) \mapsto X_t(\omega)$$

is measurable

$$\Omega \times \mathbb{R} \mapsto \mathbb{R}$$

$$F \otimes B \mapsto B$$

**Definition 6.5.** Let $\{F_t\}_{t \geq 0}$ be a filtration of $(\Omega, F, \mathbb{P})$. A stochastic process $(X_t)_{t \geq 0}$ is said to be adapted to the filtration $\{F_t\}_{t \geq 0}$ if $X_t$ is $F_t$-measurable, $\forall t \geq 0$.

**Definition 6.6.** A stochastic process $(X_t)_{t \geq 0}$ is progressively measurable if

$$(\omega, t) \mapsto X_t(\omega)$$

is measurable

$$\Omega \times [0, T] \mapsto \mathbb{R}$$

$$F_T \otimes B[0, T] \mapsto B$$

**Lemma 6.7.** If $(X_t)_{t \geq 0}$ is adapted to the filtration $F_t$ and right continuous almost surely, then $(X_t)_{t \geq 0}$ is progressively measurable w.r.t. $\{F_t\}_{t \geq 0}$. 

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Definition 6.13. The information up to the stopping time \( T \) is \( \left\{ \omega, t \mapsto X_t(\omega) \mid \omega \in \Omega, t \leq T \right\} \in \mathcal{F}_T \otimes \mathcal{B}[0,1] \) \( \Rightarrow \)

Definition 6.12. The information up to the stopping time \( T \) is \( \mathcal{F}_T = \{ A \in \mathcal{F}_T \mid A \cap \{ T \leq t \} \in \mathcal{F}_t \} \).

Definition 6.13. \( \mathcal{H} = \{ A \in \mathcal{F} \mid A \cap \{ T < t \} \in \mathcal{F}_t, \forall t \geq 0 \} \).

Lemma 6.14. If \( \mathcal{G}_t = \mathcal{F}_{t+}, \forall t \geq 0 \), then \( \mathcal{G}_T = \mathcal{H} \).
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(\{\mathcal{F}_t\}_{t \geq 0}\) be a filtration of \((\Omega, \mathcal{F}, \mathbb{P})\).

**Lemma 6.15.** i) \(T = a\) is a stopping time and \(\mathcal{F}_T = \mathcal{F}_a\).

ii) If \(T\) is a \(\{\mathcal{F}_t\}_{t \geq 0}\) stopping time, \(T\) is \(\mathcal{F}_T\) measurable.

iii) If \(S, T\) are \(\{\mathcal{F}_t\}_{t \geq 0}\) stopping times, then \(S \vee T\) and \(S \wedge T\) are stopping times.

iv) If \(S, T\) are stopping times with \(S \leq T\), then \(\mathcal{F}_S \subseteq \mathcal{F}_T\).

Proof. Take \(A \in \mathcal{F}_S\). Then we have

\[A \cap \{T \leq t\} = A \cap \{S \leq T \leq t\} = A \cap \{S \leq t\} \cap \{T \leq t\} \in \mathcal{F}_t, \forall t \geq 0\]

v) If \(S_n \uparrow S\), and each is a \(\{\mathcal{F}_t\}_{t \geq 0}\) stopping time, then \(S\) is a \(\{\mathcal{F}_t\}_{t \geq 0}\) stopping time.

Proof. \(\{S \leq t\} = \bigcap_{n \geq 1} \{S_n \leq t\} \in \mathcal{F}_t\).

vi) \(S_n \downarrow S\), each \(S_n\) is a \(\{\mathcal{F}_t\}_{t \geq 0}\) stopping time, then \(S\) is a \(\{\mathcal{F}_t\}_{t \geq 0}\) stopping time.

Proof. \(\{S \geq t\} = \bigcap_{n \geq 1} \{S_n \geq t\} \in \mathcal{F}_t\).

vii) If \(S_n \downarrow S\) and \((S_n)\) is stationary, i.e. \((S_n(\omega))_{n \geq 1}\) eventually is constant a.s., then \(S\) is a \(\{\mathcal{F}_t\}_{t \geq 0}\) stopping time.

viii) Take \(A \in \mathcal{F}\). \(T_A(\omega) = \begin{cases} T(\omega) & \text{if } \omega \in A \\ \infty & \text{otherwise} \end{cases}\)

Then \(T_A\) is a \(\{\mathcal{F}_t\}_{t \geq 0}\) stopping time i.f.f. \(A \in \mathcal{F}_T\).

Proof. \(\iff \{T_A \leq t\} = A \cap \{T \leq t\} \in \mathcal{F}_t, t \geq 0\)

\(\implies \mathcal{F}_T = \{B \in \mathcal{F} \mid B \cap \{T \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}\)

**Lemma 6.16.** Let \(T\) be a stopping time and \(S \geq T\) is \(\mathcal{F}_T\)-measurable. Then \(S\) is also a stopping time. In particular, \(T_n = \left\lfloor \frac{nT}{n} \right\rfloor\) is a \(\{\mathcal{F}_t\}_{t \geq 0}\) stopping time \(\forall n\).

Proof.

\[\{S \geq t\} = \{S \leq t, T \leq S\} = \{S \leq t\} \cap \{T \leq t\} \in \mathcal{F}_t, \forall t \geq 0\]

Suppose \((X_t)_{t \geq 0}\) is a continuous adapted process w.r.t. \(\{\mathcal{F}_t\}_{t \geq 0}\).

\[T_C = \inf_{t \geq 0} \{t \geq 0 \mid X_t \in C\}, C \subseteq \mathcal{B}(\mathbb{R})\]

**Lemma 6.17.** If \(O\) is an open set, then \(T_O\) is a \(\{\mathcal{F}_t\}_{t \geq 0}\) stopping time. If \(C\) is a closed set, then \(T_C\) is a \(\{\mathcal{F}_t\}_{t \geq 0}\) stopping time.
Proof. i) \( \{ T_O < t \} = \{ X_S \in O^c, \forall S < t \} = \{ X_S \in O^c, \forall S \in O \cap [0,t) \} \). \( O^c \) is closed. Both are \( \mathcal{F}_t \)-measurable, so \( \{ T_O < t \} \in \mathcal{F}_t \).

ii) \( \{ T_C \leq t \} = \{ \inf_{0 \leq S \leq t} d(X_S, C) = 0 \} = \{ \inf_{O \cap [0,t]} d(X_S, C) = 0 \} \). Note that \( d(X_S, C) \) is a continuous function for closed \( C \). Therefore, \( \{ T_C \leq t \} \in \mathcal{F}_t \). ■