For a GWN on \([0,1], \mathcal{B}\) with Lebesgue intensity and for \(f \in L^2[0,1]\), \(G(f)\) is a normal random variable with mean zero and variance

\[
E(G(f)^2) = \|f\|_2^2 = \int_0^1 f^2(x)dx
\]

and for \(f, g \in L^2[0,1]\), we have

\[
E(G(f)G(g)) = \int_0^1 f(x)g(x) dx.
\]

In particular, for \(f = 1_{[0,t]}, t \in [0,1]\) we have \(G(1_{[0,t]}) \sim N(0, t)\) and \(E(G(1_{[0,t]})G(1_{[0,s]})) = t \wedge s\) for \(t, s \in [0,1]\). Thus,

\[
X_t := G(1_{[0,t]}), t \in [0,1]
\]

gives a pre–SBM on \([0,1]\) when \(G\) is a GWN on \([0,1], \mathcal{B}\) with Lebesgue intensity.

### 3.0.1 Kolmogorov’s Continuity Theorem

The pre–SBM we constructed in the previous section is defined for every \(t\), but is not a continuous function, in general. We need to modify it to get a continuous version. We define the following.

**Definition 3.1.** We say that \((\hat{Y}(t))_{t \in [0,1]}\) is a modification of \((Y(t))_{t \in [0,1]}\) if

\[
P(Y(t) = \hat{Y}(t)) = 1 \text{ for all } t \in [0,1].
\]

Furthermore, \((\hat{Y}(t))_{t \in [0,1]}\) is a continuous modification of \((Y(t))_{t \in [0,1]}\) if \(t \mapsto \hat{Y}(t)\) is continuous almost surely and \(P(Y(t) = \hat{Y}(t)) = 1\) for all \(t \in [0,1]\).

To construct a continuous modification of \(X(t), t \in [0,1]\), we will use the following theorem.

**Theorem 3.2 (Kolmogorov’s continuity theorem).** Let \((X_t)_{t \in [0,1]}\) be a stochastic process such that

\[
\sup_{0 \leq s < t \leq 1} \frac{E \left| X_t - X_s \right|^\alpha}{|t - s|^{1+\delta}} \leq C
\]

for some \(C, \alpha, \delta > 0\). Then there exists a continuous modification \((\hat{X}_t)_{t \in [0,1]}\) of \((X_t)_{t \in [0,1]}\) which is a.s. \(\gamma\)-Hölder for \(\gamma \in (0, \delta/\alpha)\), i.e., for every \(\gamma \in (0, \delta/\alpha)\) the random variable

\[
\sup_{0 \leq s < t \leq 1} \frac{\left| X_t - \hat{X}_s \right|}{|t - s|^{\gamma}} < \infty \text{ almost surely.}
\]
To construct a continuous modification from a pre-SBM \((X_t)_{t \in [0,1]}\) and get SBM on \([0,1]\), we note that for any \(0 \leq t < s \leq 1\), we have

\[
\frac{E |X_t - X_s|^\alpha}{|t - s|^{{\alpha}/2}} = E |Z|^\alpha \quad \text{for all} \quad 0 \leq s < t \leq 1.
\]

since \(X_t - X_s \sim N(0, t - s)\) implies that \(E |X_t - X_s|^\alpha = (t - s)^{\alpha/2} E |Z|^\alpha\) where \(Z \sim N(0, 1)\). By Kolmogorov’s Continuity theorem with \(\alpha > 2, \delta = (\alpha - 2)/2\) we can get a continuous modification of \((X_t)_{t \in [0,1]}\) which is a.s. \(\gamma\)- Hölder with \(0 < \gamma < 1/2 - 1/\alpha\). Since \(\alpha > 2\) is arbitrary we have the proof of Theorem ??.

Next we prove Kolmogorov’s Continuity theorem (Theorem 3.2).

**Proof of Theorem 3.2.** We consider the set of dyadic rationals in \([0,1]\). We will use \(X(t)\) instead of \(X_t\), when needed. Define

\[D_n = \{k2^{-n} \mid k = 0, 1, \ldots, 2^n - 1\} \text{ for } n \geq 0 \text{ and } D = \{1\} \cup \bigcup_{n \geq 0} D_n.\]

Clearly, \(D\) is a countable dense subset of \([0,1]\). We define

\[
\Delta_n X(t) := X(t + 2^{-n}) - X(t), \quad t \in D_n, n \geq 0.
\]

**Step 1.** First we claim that for \(\gamma \in (0, \delta/\alpha)\),

\[
\sum_{n=0}^{\infty} \sum_{t \in D_n} P (|\Delta_n X(t)| \geq 2^{-\gamma n}) < \infty,
\]

which, by union bound, implies that \(\sum_{n=0}^{\infty} \sum_{t \in D_n} P (|\sup_{t \in D_n} 2^\gamma n \cdot |\Delta_n X(t)| \geq 1) < \infty\) and by first Borel-Cantelli lemma

\[
P (\sup_{t \in D_n} 2^\gamma n \cdot |\Delta_n X(t)| \geq 1 \text{ i.o.}) = 0
\]

and thus the random variable

\[
R_\gamma := \sup_{n \geq 0} \sup_{t \in D_n} 2^\gamma n \cdot |\Delta_n X(t)| < \infty \text{ almost surely.}
\]

Using Markov’s inequality and the hypothesis that \(E |X_t - X_s|^\alpha \leq C|t - s|^{1+\delta}\) for all \(t, s \in [0,1]\) for some \(C, \alpha, \delta > 0\) we can bound the LHS of (3.1) by

\[
\sum_{n=0}^{\infty} \sum_{t \in D_n} 2^{\alpha \gamma n} \cdot E |\Delta_n X(t)|^\alpha \leq C \sum_{n=0}^{\infty} 2^{\alpha \gamma n} \cdot 2^{\gamma n(1+\delta)} = C \sum_{n=0}^{\infty} 2^{-n(\delta-\alpha \gamma)} < \infty
\]

as \(\gamma < \delta/\alpha\) and this proves the claim.

**Step 2.** We claim that

\[
R'_\gamma := \sup_{s, t \in D, s < t} \frac{|X(t) - X(s)|}{|t - s|^\gamma} \leq \frac{3R_\gamma}{1 - 2^{-\gamma}}.
\]

We can ignore the case \(s = 0, t = 1, \text{ as } |X(1) - X(0)| \leq R_\gamma\). Otherwise, for \(t, s \in D, s < t\) we have \(t - s \in [2^{-k}, 2 \cdot 2^{-k})\) for some \(k \geq 1\) and there exists \(u \in D_k\) such that \(u \leq s < u + 2^{-k} \leq t < u + 3 \cdot 2^{-k}\).
If \( t \leq u + 2 \cdot 2^{-k} \), we take \( s_k = u, t_k = s_k + 2^{-k} \), otherwise we take \( s_k = u + 2^{-k}, t_k = s_k + 2^{-k} \). Note that, \( t_k - s_k = 2^{-k} \) and \( s_k \in D_k \) and thus \( |X(t_k) - X(s_k)| \leq R_\gamma \cdot 2^{-k} \). Moreover, we can choose a sequence \( s_i, t_i \in D_i, k < i \leq n \) such that \( s_n = s, t_n = t \) and \( |s_i - s_{i-1}|, |t_i - t_{i-1}| \in \{0, 2^{-1}\} \) for all \( k < i \leq n \). Since

\[
X(t) = X(t_k) + \sum_{i=k+1}^{n} (X(t_i) - X(t_{i-1})),
\]

\[
X(s) = X(s_k) + \sum_{i=k+1}^{n} (X(s_i) - X(s_{i-1})),
\]

we have

\[
|X(t) - X(s)| \leq R_\gamma \left( 2^{-\gamma k} + 2 \sum_{i=k+1}^{n} 2^{-\gamma i} \right) \leq \frac{32^{-\gamma k}}{1 - 2^{-\gamma}} \cdot R_\gamma \leq \frac{3R_\gamma}{1 - 2^{-\gamma}} \cdot |t - s|^{\gamma}.
\]

**Step 3.** Fix \( \gamma \in (0, \delta/\alpha) \). By equation (3.3), the event \( A = \{R_\gamma < \infty\} \) has probability 1. Fix \( \omega \in A \). Now we claim that for \( t \in [0, 1] \setminus D \) and any sequence \( t_n \to t, t_n \in D \), \( \lim_{n \to \infty} X(t_n) \) exists and does not depend on the particular choice of the sequence \( (t_n)_{n \geq 1} \).

The proof follows from the fact \( |X(t_n) - X(t_m)| \leq R_\gamma |t_n - t_m|^{\gamma} \) for all \( m, n \) and thus the sequence \( X(t_n) \) is Cauchy and has a limit. The uniqueness of the limit follows from the same argument.

**Step 4.** We define

\[
\hat{X}(t) = \begin{cases} 
X(t) & \text{if } t \in D \\
\lim_{n \to \infty} X(t_n) & \text{if } t \in [0, 1] \setminus D, t_n \to t \text{ with } t_n \in D.
\end{cases}
\]

We claim that \( \hat{X}(t), t \in [0, 1] \) is a.s. continuous and \( \gamma \)-Hölder for \( \gamma \in (0, \delta/\alpha) \). Moreover, \( \hat{X}(t) = X(t) \) a.s. for \( t \in [0, 1] \).

It is easy to see that

\[
\sup_{0 \leq s < t \leq 1} \frac{|\hat{X}(t) - \hat{X}(s)|}{|t - s|^\gamma} \leq \frac{3R_\gamma}{1 - 2^{-\gamma}} < \infty \text{ a.s.}
\]

by following the same argument as in step 3. In particular, \( \hat{X}(t), t \in [0, 1] \) is a.s. continuous and \( \gamma \)-Hölder for \( \gamma \in (0, \delta/\alpha) \). Moreover, \( \hat{X}(t) = X(t) \) a.s. for \( t \in D \) and if \( t_n \to t, t_n \in D \) then \( \hat{X}(t_n) \to \hat{X}(t) \) a.s. by definition, \( X(t_n) \to X(t) \) a.s. by the same argument with \( D \) replaced by \( D \cup \{t\} \), thus \( \hat{X}(t) = X(t) \) a.s. for \( t \in [0, 1] \). \( \blacksquare \)

### 3.0.2 Brownian motion on \([0, \infty)\)

In the previous sections we have constructed SBM on \([0, 1]\). Using similar argument, we can construct countable many i.i.d. SBM’s \((B_t^{(i)})_{t \in [0,1]}, i = 0, 1, \ldots\). Define

\[
B_t := \sum_{i=0}^{[t]} B_t^{(i)} + B_{t-[t]+1} \quad \text{for } t \geq 0.
\]

It is easy to check that \( t \mapsto B_t \) is continuous on \([0, \infty)\) a.s. and it is a SBM on \([0, \infty)\) from the variance-covariance computation.