Construction of Pre-Brownian Motion

It is well-known that if $Y_1$ is a normal random variable with $N(m_1, \sigma_1^2)$ distribution, $Y_2$ is a Gaussian random variable with $N(m_2, \sigma_2^2)$ distribution, and $Y_1$ and $Y_2$ are independent, then $Y_1 + Y_2$ is a normal random variable with $N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$ distribution. This directly follows from the exact form of the characteristic function for normal distribution.

In particular, if $X_i \sim N(\mu_i, \sigma_i^2)$ are independent for $i \geq 1$, then

$$
\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).
$$

The following result will be useful to show that normal distribution is closed under $L^2$ limit.

**Proposition 2.1 (Normal distribution closed under $L^2$ limit).** Let $(X_n)_{n \geq 1}$ be a sequence of real-valued random variables such that, for every $n \geq 1$, $X_n$ follows the $N(m_n, \sigma_n^2)$ distribution. Suppose that $X_n$ converges in $L^2$ to $X$. Then

(i) The random variable $X \sim N(m, \sigma^2)$ distribution, with $m := \lim_{n \to \infty} m_n$ and $\sigma^2 := \lim_{n \to \infty} \sigma_n^2$;

(ii) the convergence also holds in $L^p$ for all $p \in [1, \infty)$.

**Proof.** (i) The convergence in $L^2$ implies that $m_n = \text{E}(X_n) \to \text{E}(X)$ and $\sigma_n^2 = \text{Var}(X_n) \to \text{Var}(X)$. Then setting $m = \text{E}(X)$ and $\sigma^2 = \text{Var}(X)$, we get, for any $t \in \mathbb{R}$,

$$
\text{E}(e^{itX}) = \lim_{n \to \infty} \text{E}(e^{itX_n}) = \lim_{n \to \infty} e^{im_n t - \sigma_n^2 t^2/2} = e^{imt - \sigma^2 t^2/2}.
$$

Thus $X$ is a Gaussian random variable with the $N(m, \sigma^2)$ distribution.

(ii) Since $X_n \overset{d}{=} \sigma_n \eta + m_n$ with $\eta$ a standard normal random variable, and since $(m_n)_{n \geq 1}$ and $(\sigma_n^2)_{n \geq 1}$ are bounded, we have that, for any $q \geq 1$,

$$
\sup_n \text{E} |X_n|^q < \infty.
$$

It follows that, for any $q \geq 1$,

$$
\sup_n \text{E} |X_n - X|^q < \infty.
$$

Let $p \in [1, \infty)$. The sequence $Y_n = |X_n - X|^p$ converges to 0 in probability and is uniformly integrable because it is bounded in $L^2$. It follows that this sequence converges to 0 in $L^1$, which is the desired result. \qed
2.1 Existence of Brownian Motion

Theorem 2.2 (Existence of SBM (Wiener 1923)). *Standard Brownian Motion* \((B_t)_{t \geq 0}\) exists.

The proof consists of three steps:

(i) constructing pre–SBM on \([0, 1]\),

(ii) finding a continuous modification to construct a SBM on \([0, 1]\).

(iii) constructing SBM on \([0, \infty) = \bigcup_{k=0}^{\infty} [k, k + 1)\).

**Theorem 2.3.** Standard Brownian Motion \((B_t)_{t \in [0,1]}\) exists.

2.1.1 Gaussian White Noise and Construction of pre–SBM on \([0, 1]\)

First we define a centered Gaussian space and centered Gaussian process.

**Definition 2.4.**

(i) A **centered Gaussian space** is a closed linear subspace of \(L^2(\Omega, \mathcal{F}, P)\) which contains only centered Gaussian random variables.

(ii) A real-valued random process \((X_t)_{t \in I}\) is called a **centered Gaussian process** on \(I\) if any finite linear combination of the variables \(X_t, t \in I\) is a centered Gaussian/normal random variable.

One can prove that, for a centered Gaussian process \((X_t)_{t \in I}\) on \(I\), the closed linear subspace of \(L^2\) spanned by the variables \(X_t, t \in I\), is a centered Gaussian space, which is called the Gaussian space generated by the process \(X\). In fact, a centered Gaussian process \((X_t)_{t \in I}\) is uniquely characterized by the covariance function

\[
K(s, t) := \text{Cov}(X_s, X_t) = E(X_s X_t) \text{ for } s, t \in I.
\]

Note that, pre–SBM is nothing but a centered Gaussian process with covariance function

\[
K(s, t) = s \wedge t.
\]

**Definition 2.5.** Let \((E, \mathcal{E})\) be a measure space and \(\mu\) be a \(\sigma\)-finite measure on \((E, \mathcal{E})\). A **Gaussian white noise (GWN)** on \((E, \mathcal{E})\) with intensity \(\mu\) is an isometry \(G\) from \(L^2(E, \mathcal{E}, \mu)\) into a centered Gaussian space.

**Proposition 2.6** (Existence of Gaussian White Noise). Suppose there exists a Uniform\((0, 1)\) random variable defined on \((\Omega, \mathcal{F}, P)\). Then, on \((\Omega, \mathcal{F}, P)\) there exists a Gaussian white noise on \(([0, 1], \mathcal{B})\) with Lebesgue intensity \(\lambda\).

**Remark 2.7.** The condition “there exists a Uniform\((0, 1)\) random variable defined on \((\Omega, \mathcal{F}, P)\)” implies that there is a countable sequence of i.i.d. standard normal r.v.s defined on \((\Omega, \mathcal{F}, P)\). The same proof works for constructing GWN on \((E, \mathcal{E})\) with intensity \(\mu\) if \(L^2(E, \mathcal{E}, \mu)\) is separable, i.e., has a countable orthonormal basis. In general, \(L^2(E, \mathcal{E}, \mu)\) is a Hilbert space and always has an orthonormal basis \((f_\lambda)_{\lambda \in \Lambda}\) (\(\Lambda\) need not be countable). As long as we have a collection of i.i.d. standard normal r.v.s \((X_\lambda)_{\lambda \in \Lambda}\) defined on \((\Omega, \mathcal{F}, P)\) we can construct GWN on \((E, \mathcal{E})\) with intensity \(\mu\).
Proof. An isometry, between two inner product spaces must map an orthonormal basis to an orthonormal set.

Let \((f_i)_{i=1,2,...}\) be an orthonormal basis of \(H \coloneqq L^2([0,1], \mathcal{B}, \lambda)\). For \(f \in H\), we have

\[
f = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i
\]

with \(\sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 = \|f\|_2^2 < \infty\).

Under the assumption on \((\Omega, \mathcal{F}, \mathbb{P})\), we can construct a countably infinite collection \((\eta_i)_{i \geq 1}\) of i.i.d. \(N(0,1)\) rvs on \((\Omega, \mathcal{F}, \mathbb{P})\). We define,

\[
G(f) := \sum_{i=1}^{\infty} \langle f, f_i \rangle \eta_i, \quad \text{for } f \in H.
\]

Note that, \(G(f)\) is well defined in \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) as \((\eta_i)_{i \geq 1}\) is an orthonormal system (or use the \(L^2\) convergence theorem or its generalization Kolmogorov’s three series theorem). Clearly \(G\) takes values in the Gaussian space generated by \((\eta_i)_{i \geq 1}\). Furthermore, \(G\) is an isometry since it maps the orthonormal basis \((f_i)_{i \geq 1}\) to an orthonormal system.

Now, for a GWN on \([0,1], \mathcal{B}\) with Lebesgue intensity and for \(f \in L^2[0,1]\), \(G(f)\) is a normal random variable with mean zero and variance

\[
\mathbb{E}(G(f)^2) = \|f\|_2^2 = \int_0^1 f^2(x)dx
\]

and for \(f, g \in L^2[0,1]\), we have

\[
\mathbb{E}(G(f)G(g)) = \int_0^1 f(x)g(x) \, dx.
\]

In particular, for \(f = \mathbb{1}_{[0,t]}, t \in [0,1]\) we have \(G(\mathbb{1}_{[0,t]}) \sim N(0,t)\) and \(\mathbb{E}(G(\mathbb{1}_{[0,t]}))G(\mathbb{1}_{[0,s]})) = t \wedge s\) for \(t, s \in [0,1]\). Thus,

\[
X_t := G(\mathbb{1}_{[0,t]}), \quad t \in [0,1]
\]

gives a pre–SBM on \([0,1]\) when \(G\) is a GWN on \(([0,1], \mathcal{B})\) with Lebesgue intensity.

### 2.1.2 Kolmogorov’s Continuity Theorem

The pre–SBM we constructed in the previous section in defined for every \(t\), but is not a continuous function, in general. We need to modify it to get a continuous version. We define the following.

**Definition 2.8.** We say that \((\hat{Y}(t))_{t \in [0,1]}\) is a modification of \((Y(t))_{t \in [0,1]}\) if

\[
\mathbb{P}(Y(t) = \hat{Y}(t)) = 1 \quad \text{for all } t \in [0,1].
\]

Furthermore, \((\hat{Y}(t))_{t \in [0,1]}\) is a continuous modification of \((Y(t))_{t \in [0,1]}\) if \(t \mapsto \hat{Y}(t)\) is continuous almost surely and \(\mathbb{P}(Y(t) = \hat{Y}(t)) = 1 \quad \text{for all } t \in [0,1]\).

To construct a continuous modification of \(X(t), t \in [0,1]\), we will use the following theorem.
Theorem 2.9 (Kolmogorov’s continuity theorem). Let \((X_t)_{t \in [0,1]}\) be a stochastic process such that
\[
\sup_{0 \leq s < t \leq 1} \frac{E|X_t - X_s|^\alpha}{|t - s|^{1+\delta}} \leq C
\]
for some \(C, \alpha, \delta > 0\). Then there exists a continuous modification \((\hat{X}_t)_{t \in [0,1]}\) of \((X_t)_{t \in [0,1]}\) which is a.s. \(\gamma\)-Hölder for \(\gamma \in (0, \delta/\alpha)\), i.e., for every \(\gamma \in (0, \delta/\alpha)\) the random variable
\[
\sup_{0 \leq s < t \leq 1} \frac{|\hat{X}_t - \hat{X}_s|}{|t - s|^\gamma} < \infty \text{ almost surely.}
\]

To construct a continuous modification from a pre–SBM \((X_t)_{t \in [0,1]}\) and get SBM on \([0,1]\), we note that for any \(0 \leq t < s \leq 1\), we have
\[
\frac{E|X_t - X_s|^\alpha}{|t - s|^{\alpha/2}} = E|Z|^\alpha \text{ for all } 0 \leq s < t \leq 1.
\]
since \(X_t - X_s \sim N(0, t - s)\) implies that \(E|X_t - X_s|^\alpha = (t - s)^{\alpha/2} E|Z|^\alpha\) where \(Z \sim N(0,1)\). By Kolmogorov’s Continuity theorem with \(\alpha > 2, \delta = (\alpha - 2)/2\) we can get a continuous modification of \((X_t)_{t \in [0,1]}\) which is a.s. \(\gamma\)-Hölder with \(0 < \gamma < 1/2 - 1/\alpha\). Since \(\alpha > 2\) is arbitrary we have the proof of Theorem 2.3.

Next we prove Kolmogorov’s Continuity theorem (Theorem 2.9).