Homework 1

MATH 562: Theory of Probability II

Due date: September 09, 2021

Please indicate whom you worked with. It will not affect your grade in any way. The first problem is worth 40 points, and the second problem is worth 10 points.

1. (Lévy’s construction of BM)
   (a) We consider the set of dyadic rationals in $[0, 1]$, i.e., $\mathcal{D}_n := \{k/2^n \mid 0 \leq k < 2^n\}$ for $n \geq 0$ and $\mathcal{D} := \{1\} \cup \bigcup_{n \geq 0} \mathcal{D}_n$. Let $(Z_d, d \in \mathcal{D})$ be a sequence of i.i.d. $N(0, 1)$ random variables.

   Using the fact that for $Z \sim N(0, 1)$ we have $\mathbb{P}(\{|Z| \geq x\} \leq 2e^{-x^2/2}$ for $x > 0$ and Borel-Catelli lemma, show that
   
   $$\mathbb{P}\left(\max_{d \in \mathcal{D} \setminus \mathcal{D}_{n-1}} |Z_d| \geq c\sqrt{n} \text{ i.o.}\right) = 0$$

   for any fixed $c > \sqrt{2 \log 2}$.

   (b) Next, we define $\varphi_0(t) := t$ for $t \in [0, 1]$ and for $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}, n \geq 1$ we define the functions
   
   $$\hat{\varphi}_d(t) := \begin{cases} 1 - 2^n|t - d| & \text{if } |t - d| \leq 2^{-n} \\ 0 & \text{otherwise.} \end{cases}$$

   Note that, for fixed $n \geq 1$, the functions $\hat{\varphi}_d$, $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ have disjoint support and is uniformly bounded by 1. Define
   
   $$\varphi_d(\cdot) := 2^{-(n+1)/2}\hat{\varphi}_d(\cdot), \quad \text{for } d \in \bigcup_{n \geq 1} \mathcal{D}_n.$$ 

   One can check that the functions $\{\varphi_d(\cdot) \mid d \in \bigcup_{n \geq 0} \mathcal{D}_n\}$ form an orthonormal basis in the Dirichlet space $D[0, 1] = \{f \mid f \text{ is weakly differentiable with } f(0) = 0 \text{ and } \int_0^1 |f'(t)|^2dt < \infty\}$ under the inner product
   
   $$\langle f, g \rangle_D := \int_0^1 f'(t)g'(t)dt.$$ 

   Show that
   
   $$\varphi_d(t) = \langle \varphi_d(\cdot), \min(t, \cdot) \rangle_D$$

   and by Parseval’s identity $\sum_{d \in \mathcal{D}} \varphi_d(t)^2 = t$ for all $t \in [0, 1]$.

   (c) We define a sequence of random continuous functions as follows:
   
   $$X_n(\cdot) := \sum_{d \in \mathcal{D}_n} Z_d\varphi_d(\cdot) \text{ for } n \geq 0.$$ 

   Using (a), show that the functions $(X_n)_{n \geq 0}$ are almost surely Cauchy in $C([0, 1])$ w.r.t. the $\|\cdot\|_\infty$ norm. Thus, it converges to a continuous function $X_\infty$ uniformly.

   (d) Note that, for $d \in \mathcal{D}_n$, $X_m(d) = X_n(d)$ for all $m \geq n$. By checking the covariance structure at the dyadic rational points, i.e., $\text{Cov}(X_\infty(d), X_\infty(d'))$ for $d, d' \in \mathcal{D}$, argue that $X_\infty(t), t \in [0, 1]$ is a SBM.

| Solution: | (a) Clearly, $|\mathcal{D}_n \setminus \mathcal{D}_{n-1}| = 2^{n-1}$ and thus

   $$\mathbb{P}\left(\max_{d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}} |Z_d| \geq c\sqrt{n}\right) \leq |\mathcal{D}_n \setminus \mathcal{D}_{n-1}| \cdot \mathbb{P}(\{|Z| \geq c\sqrt{n}\} \leq 2^n e^{-c^2 n/2} = e^{-(c^2 - 2 \ln 2)/2}.\right)$$

   In particular, for $c < \sqrt{2 \ln 2}$, we have

   $$\sum_{n \geq 1} \mathbb{P}(\max_{d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}} |Z_d| \geq c\sqrt{n}) < \infty$$ |
and by Borel Cantelli Lemma we have \( P(\max_{d \in D_n \setminus D_{n-1}} |Z_d| \geq c \sqrt{n} \, \text{i.o.}) = 0. \)

(b) The first equality follows since \( f(t) = \int_0^1 f'(s)1_{[a,b]}(s)ds \) and \( \min(x, t) = \int_0^t 1_{[a,b]}(s)ds \). The second equality follows as \( \langle \min(\cdot, t), \min(\cdot, s) \rangle_D = \min\{t, s\} \) and by Parseval’s identity we have

\[
\sum_{d \in D} \varphi_d(t)^2 = \sum_{d \in D} \langle \varphi_d(\cdot), \min(\cdot, t) \rangle_D^2 = \langle \min(\cdot, t), \min(\cdot, t) \rangle_D = t.
\]

(c) We have

\[
\|X_n - X_{n-1}\| = \left\| \sum_{d \in D_n \setminus D_{n-1}} Z_d \varphi_d \right\|_\infty \leq 2^{-(n+1)/2} \max_{d \in D_n \setminus D_{n-1}} |Z_d|.
\]

By (a), we have \( R := \sup_{n \geq 1} n^{-1/2} \max_{d \in D_n \setminus D_{n-1}} |Z_d| < \infty \) a.s. Thus

\[
\max_{m \geq 1} \|X_{n+m} - X_n\| \leq \max_{m \geq 1} \sum_{k=n+1}^{n+m} 2^{-(k+1)/2} R \sqrt{k} \leq R \sum_{k=n+1}^\infty 2^{-(k+1)/2} \cdot \sqrt{k} \to 0 \text{ as } n \to \infty.
\]

This proves the Cauchy property and by completeness of \( C[0,1] \) w.r.t. the \( \| \cdot \|_\infty \) norm we have the result.

(d) For every \( t \), \( X_n(t) \to X_\infty(t) \) in both a.s. and \( L^2 \) sense. Note that, for \( d \in D_n \), \( X_m(d) = X_n(d) \) for all \( m \geq n \). Thus, for \( d, d' \in D_n \), we have

\[
\text{Cov}(X_\infty(d), X_\infty(d')) = \text{Cov}(X_n(d), X_n(d')) = \sum_{d \in D_n} \varphi_d(d) \varphi_d(d') = \sum_{d \in D} \varphi_d(d) \varphi_d(d') = \sum_{d \in D} \langle \varphi_d(\cdot), \min(\cdot, d) \rangle_D \langle \varphi_d(\cdot), \min(\cdot, d') \rangle_D = \langle \min(\cdot, d), \min(\cdot, d') \rangle_D = \min(d, d').
\]

The covariance structure for general \( s, t \) follows by taking limit through dyadic rationals.

2. (Non-smoothness of BM path) Fix \( \gamma \in (1/2, 1] \). For \( t \geq 0 \), define

\[
H_\gamma(t) := \sup_{h \in [0,1]} \frac{|B(t+h) - B(t)|}{h^\gamma}.
\]

Show that,

\[
P(H_\gamma(t) = \infty \text{ for all } t \in [0,1]) = 1.
\]

i.e., BM is NOT \( \gamma \)-Hölder at any point a.s., if \( \gamma > 1/2 \).

**Hint:** Note that, if \( H_\gamma(t) \leq M \) for some \( M < \infty \), \( t \in [0,1] \) and \( t \in ((k-1)/2^n, k/2^n) \), then

\[
|B(\ell/2^n) - B((\ell - 1)/2^n)| \leq M \cdot ((\ell + 1 - k) \gamma + (\ell - k) \gamma) \cdot 2^{-n\gamma} \leq 2M(m+1) \gamma \cdot 2^{-n\gamma}
\]

for any \( k+1 \leq \ell \leq k+m \). Use the fact that for \( Z \sim N(0,1) \) we have \( P(|Z| \leq t) \leq t \) for \( t > 0 \). Finally choose \( m \) such that \( m(\gamma - 1/2) > 1 \) and use Borel Cantelli lemma.

**Solution:** Using independence for disjoint increments and the fact that for \( Z \sim N(0,1) \) we have \( P(|Z| \leq t) \leq t \) for \( t > 0 \), for any fixed integer \( m \geq 1 \) and constant \( c < 0 \), we have

\[
P(\sup_{k+1 \leq \ell \leq k+m} |B(\ell/2^n) - B((\ell - 1)/2^n)| \leq c \cdot 2^{-n\gamma}) = \prod_{\ell=k+1}^{k+m} P(2^{n/2} |B(\ell/2^n) - B((\ell - 1)/2^n)| \leq c \cdot 2^{-n(\gamma - 1/2)}) \leq c^m \cdot 2^{-nm(\gamma - 1/2)}.
\]
Choose $m$ such that $m(\gamma - 1/2) > 1$. By Borel-Cantelli lemma (i) we have

$$P\left(\bigcup_{k=0}^{2^n-1} \{ \sup_{k+1 \leq \ell \leq k+m} |B(\ell/2^n) - B((\ell - 1)/2^n)| \leq c \cdot 2^{-n\gamma} \} \ i.o. \right) = 0,$$

i.e., almost surely for all $n$ large enough and for all $k = 0, 1, \ldots, 2^n - 1$, we have

$$\sup_{k+1 \leq \ell \leq k+m} |B(\ell/2^n) - B((\ell - 1)/2^n)| > c \cdot 2^{-n\gamma}.$$

By the hint, taking $c = 2M(m + 1)^\gamma$, we get that

$$P(H_\gamma(t) > M \text{ for all } t \in [0, 1]) = 1.$$

Since $M$ is arbitrary, taking $M \uparrow \infty$ we get the result.