You should finish this final on your own, be rigorous and precise.

1. (10 points) Let \( (\Omega, \mathcal{F}, P) \) be a probability space and \( (\mathcal{G}_t)_{t \geq 0} \) be a filtration of \( \mathcal{F} \). Define \( \mathcal{G}^+_t := \bigcap_{s \geq 0} \tilde{\mathcal{G}}_s \) a right continuous filtration such that \( \tilde{\mathcal{G}}_s \subseteq \tilde{\mathcal{G}}_r \) for all \( s \geq 0 \).

Show that \( (\mathcal{G}^+_t)_{t \geq 0} \) is a right continuous filtration.

2. (10 points) If \( K \) is a locally bounded continuous adapted process and \( X \) is a continuous semimartingale, prove that, as \( \varepsilon \downarrow 0 \)

\[
\frac{1}{\varepsilon} \int_0^1 K_s(X_{s+\varepsilon} - X_s) \, ds \to \int_0^1 K_s dX_s \quad \text{in probability}
\]

and

\[
\frac{1}{\varepsilon} \int_0^1 K_s(X_{s+\varepsilon} - X_s)^2 \, ds \to \int_0^1 K_s d\langle X, X \rangle_s \quad \text{in probability}.
\]

3. (10 points) For any two continuous semimartingales \( X, Y \) the Stratonovich integral of \( Y \) with respect to \( X \) is defined as

\[
\int_0^t Y_s \circ dX_s := \int_0^t Y_s dX_s + \frac{1}{2} \langle Y, X \rangle_t, \quad t \geq 0.
\]

Suppose that \( X_t = (X^1_t, X^2_t, \ldots, X^d_t), t \geq 0 \) is a \( \mathbb{R}^d \)-valued of continuous semimartingale and \( f : \mathbb{R}^d \to \mathbb{R} \) is a thrice continuously differentiable function. Show that

\[
f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(X_s) \circ dX_s^{(i)}, \quad t \geq 0.
\]

4. (10 points) Let \( X_t = (X^1_t, X^2_t) \) be the unique solution to the SDE

\[
dX^1_t = 2dt + dB^1_t + dB^2_t, \quad 0 \leq t \leq T
\]

\[
dX^2_t = 6dt + dB^1_t - dB^2_t, \quad 0 \leq t \leq T
\]

\[
X_0 = x_0 \in \mathbb{R}^2,
\]

on some filtered probability space \( (\Omega, \mathcal{F}, P, \{ \mathcal{F}_t \}_{0 \leq t \leq T}) \) with a SBM \( B_t = (B^1_t, B^2_t) \). Find a probability measure \( Q \) on \( \mathcal{F}_T \) (i.e., its Radon-Nikodym derivative w.r.t. \( P \)), under which \( (X_t)_{0 \leq t \leq T} \) is a \( (\mathcal{F}_t)_{0 \leq t \leq T} \) martingale.

5. (10 points) Let \( B \) be a one-dimensional \( (\mathcal{F}_t)_{t \geq 0} \) Brownian motion defined on a filtered probability space. Let \( \mu_t \) and \( \sigma_t \) be two uniformly bounded, \( (\mathcal{F}_t)_{t \geq 0} \) progressively measurable processes in \( \mathbb{R} \). By using Itô’s formula, find a continuous semimartingale \( X_t \) which satisfies

\[
X_t = 1 + \int_0^t X_s \mu_s \, ds + \int_0^t X_s \sigma_s \, dB_s, \quad t \geq 0.
\]

Moreover, argue that \( X \) is unique.