

## Characteristic functions

### 9.1 Helly's Selection Theorem

#### 9.1.1 Extended Random Variables

**Definition 9.1.** An extended random variable is a measurable function  $X : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathbb{R}^*, \mathcal{B}^*)$  where  $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ ,  $\mathcal{B}^* = \sigma(\mathcal{B}, \{\infty\}, \{-\infty\})$  such that

- $\mathbb{P}(X = \infty) = \lim_{x \uparrow \infty} \mathbb{P}(X > x)$  and
- $\mathbb{P}(X = -\infty) = \lim_{x \downarrow -\infty} \mathbb{P}(X \leq x)$ .

**Definition 9.2.** An extended distribution function (EDF) is a non-decreasing, right-continuous function from  $\mathbb{R}$  to  $[0, 1]$ .

Note that for an EDF  $F$ , it is a CDF iff  $1 - F(x) + F(-x) \rightarrow 0$  as  $x \uparrow \infty$ . Convergence in distribution generalizes naturally to extended random variables:

**Definition 9.3.** Let  $X_n, X$  be extended random variables with EDF's  $F_n$  and  $F$ , respectively. We say  $X_n$  converges in distribution to  $X$  and write  $X_n \xrightarrow{(d)} X$ , if

$$F_n(x) \rightarrow F(x)$$

as  $n \rightarrow \infty$  for every continuity point  $x$  of  $F$ .

#### 9.1.2 Helly's Selection Theorem

**Theorem 9.4 (Helly Bray Selection theorem).** Given a sequence of EDF's  $F_1, F_2, \dots$  there exists a subsequence  $(n_k)$  such that  $F_{n_k} \xrightarrow{(d)} F$  for some EDF  $F$ .

To prove this theorem, we need the following lemma:

**Lemma 9.5.** Let  $(F_n)_{n \geq 1}$  be a sequence of EDFs such that for a dense subset  $D$ ,  $\lim_{n \rightarrow \infty} F_n(d) = G(d)$  exists for all  $d \in D$ . Define  $F_*(x) = \inf_{d > x} G(d)$ , then  $F_n(x) \xrightarrow{(d)} F_*$ .

*Proof.* It is easy to check that  $F_*$  is an EDF. Then, for any continuity point  $x$  of  $F_*$ , there exists  $d_1, d_2 \in D$  such that  $d_1 < x < d_2$  and  $F_*(x) - \varepsilon < F_n(d_1) \leq F_n(x) \leq F_n(d_2) < F_*(x) + \varepsilon$  for  $n$  large enough. ■

*Proof of Theorem 9.4.* Use Cantor's Diagonal Argument to find a sequence  $F_{n_k}$  which is convergent at any point within  $\mathbb{Q}$ , then by Lemma 9.5 we are done. ■

**Definition 9.6.** A collection of distributions  $\{\mathbb{P}_\lambda : \lambda \in \Lambda\}$  on  $\mathbb{R}$  is tight if

$$\lim_{x \rightarrow \infty} \sup_{\lambda \in \Lambda} \mathbb{P}_\lambda((-x, x)^c) = 0.$$

Equivalently,  $\forall \varepsilon > 0, \exists$  a compact set  $B_\varepsilon$  such that  $\mathbb{P}_\lambda(B_\varepsilon^c) < \varepsilon, \forall \lambda \in \Lambda$ .

**Theorem 9.7 (Helly's selection theorem).** Let  $(F_n)_{n \geq 1}$  be a sequence of CDFs which are tight, then there exists a subsequence  $(n_k)$  such that  $F_{n_k} \xrightarrow{(d)} F$  for some CDF  $F$ .

*Proof.* By Helly Bray Selection Theorem,  $F_{n_k} \xrightarrow{(d)} F_*$  for some EDF  $F_*$ . Given  $\varepsilon > 0$ , find two continuity points of  $F_*$ ,  $\pm d$ , such that  $\sup_k |1 - F_{n_k}(d) + F_{n_k}(-d)| \leq \varepsilon$ , then  $|1 - F_*(d) + F_*(-d)| \leq \varepsilon$ . Thus  $F_*$  is a CDF. ■

## 9.2 Metric on the space of Probability Measures

Take  $\{f_n, n \geq 1\}$  a countable sequence of continuous functions bounded by 1, which is dense in  $C_b(\mathbb{R})$ . Define

$$d(\mu, \gamma) := \sum_{k \leq 1} \frac{1}{2^k} \left| \int f_k d\mu - \int f_k d\gamma \right|,$$

then  $d(\mu_n, \mu) \rightarrow 0$  iff  $\mu_n \xrightarrow{(d)} \mu$ .

**Definition 9.8 (Lévy metric).** Let  $F$  and  $G$  be two CDFs, define

$$d_L(F, G) := \inf\{\varepsilon : F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon\},$$

then  $d_L(F, G) \rightarrow 0$  iff  $F_n \xrightarrow{(d)} F$ .

Thus the space of probability measures on  $\mathbb{R}$  with convergence in distribution is metrizable and is complete by Helly's selection theorem.

**Definition 9.9 (Determining Class).** Let  $\mathcal{C}$  be a collection of bounded continuous functions.  $\mathcal{C}$  is called determining class if  $\int f d\mu = \int f d\gamma \forall f \in \mathcal{C} \implies \mu = \gamma$ .

**Example 9.10.** Define

$$f_{x,\varepsilon}(y) = \begin{cases} 1 & y \leq x \\ 0 & y \geq x + \varepsilon \\ 1 - (y - x)/\varepsilon & x < y < x + \varepsilon \end{cases}$$

Then  $\mathcal{C} := \{f_{x,1/k} : x \in \mathbb{Q}, k \leq 1\}$  is a determining class, since for any CDF  $F, x \in \mathbb{Q}$ , we have  $F(x) = \lim_{k \rightarrow \infty} \int f_{x,1/k}(y) dF(y)$ .

**Example 9.11.**  $\{x \mapsto \sin(\theta x), x \mapsto \cos(\theta x), \theta \in \mathbb{R}\}$  is a determining class.

**Theorem 9.12.** Let  $\mathcal{C}$  be a determining class and  $(F_n)_{n \geq 1}$  be a tight sequence of CDFs. If  $\int g(x) dF_n(x)$  converges for  $\forall g \in \mathcal{C}$ , then  $F_n \xrightarrow{(d)} F$  for some CDF  $F$  and

$$\lim_{n \rightarrow \infty} \int g(x) dF_n(x) = \int g(x) dF(x).$$

To prove Theorem 9.12, we need the following lemma.

**Lemma 9.13.** *Let  $(S, d)$  be a metric space and  $x_n$  be a sequence in  $S$ , then  $x_n \rightarrow x$  iff  $\forall$  subsequence  $(n_k), \exists$  a further subsequence  $(m_k)$  such that  $x_{m_k} \rightarrow x$ .*

*Proof.* Only if part is easy. For the if part, assume that  $x_n \not\rightarrow x$ , then  $\exists \varepsilon > 0, \exists$  a subsequence such that  $d(x_{n_k}, x) > \varepsilon, \forall k$ . Contradiction. ■

*Proof of Theorem 9.12.* Now, we apply Lemma 9.13 to prove Theorem 9.12. By Helly Selection Theorem,  $\exists$  a subsequence  $(n_k)$  such that  $F_{n_k} \xrightarrow{(d)} F$  for some CDF  $F$  and thus

$$\int g(x)dF(x) = \lim_{n \rightarrow \infty} \int g(x)dF_n(x), \forall g \in \mathcal{C}.$$

If  $F_{m_k} \xrightarrow{(d)} G$  for some subsequence  $m_k$ , then  $\int g(x)dG(x) = \lim_{n \rightarrow \infty} \int g(x)dF_n(x), \forall g \in \mathcal{C}$ . Then  $F = G$  since  $\mathcal{C}$  is a determining class. By Lemma 9.13,  $F_n \xrightarrow{(d)} F$  and  $\lim_{n \rightarrow \infty} \int g(x)dF_n(x) = \int g(x)dF(x)$ . ■

Consider all random variables on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , namely,  $\mathcal{X} := \{X : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}) \mid X \text{ measurable}\}$ . We have metrizable spaces  $L^p(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{X}$  for  $p \geq 1$ , which guarantee the convergence of r.v.s in probability, *i.e.*, convergence w.r.t. the norm of the spaces implies the convergence in probability. We can also consider the metric  $d_{\mathbb{P}}(X, Y) := \mathbb{E} \min\{1, |X - Y|\}$  which induces convergence in probability. However, in general almost sure convergence is not metrizable. Using Lemma 9.13 one can prove that, if almost sure convergence is metrizable, it will be equivalent to convergence in probability. However, Egorov's theorem states that a.s. convergence implies almost uniform convergence.

For convergence in distribution, we have the similar metrizable spaces guaranteeing the convergence. Let  $\mathcal{M}(\Omega, \mathcal{F}) = \{\mathbb{P} \mid \mathbb{P} \text{ is a probability measure on } (\Omega, \mathcal{F})\}$ . We have seen Lévy metric, Total Variation metric, Kolmogorov-Smirnov metric and Kantarovich-Wasserstein metric for convergence in distribution on  $(\mathbb{R}, \mathcal{B})$ . In general, let  $\mathcal{D}$  be a determining class of bounded continuous functions, then we can define distance

$$d_{\mathcal{D}}(\mu, \nu) = \sup_{f \in \mathcal{D}} \left| \int f d\mu - \int f d\nu \right|.$$

Then, the metric mentioned above could be defined with corresponding determining classes:

- Total Variation metric:  $\mathcal{D} = \{f \in C_b(\mathbb{R}) \mid |f|_{\infty} \leq 1\}$ ,
- Kolmogorov-Smirnov metric:  $\mathcal{D} = \{\mathbb{1}_{(-\infty, x]} \mid x \in \mathbb{R}\}$ ,
- Kantarovich-Wasserstein metric:  $\mathcal{D} = \{f \in C_b^1 \mid |f'|_{\infty} \leq 1\}$ .

Recall that  $\mathcal{M} = \mathcal{M}(\mathbb{R}, \mathcal{B})$  with convergence in distribution is metrizable under Lévy metric and is complete by Helly's selection theorem. In general, all the other metrics mentioned are stronger than Lévy metric.

### 9.3 The Characteristic Function

**Definition 9.14.** *For a random variable  $X$ , define the **characteristic function** of  $X$ ,  $\varphi_X : \mathbb{R} \rightarrow \mathbb{C}$ , by*

$$\varphi_X(t) := \mathbb{E}(e^{itX}) = \mathbb{E}(\cos tX) + i \mathbb{E}(\sin tX), \quad t \in \mathbb{R}.$$

**Proposition 9.15.** *Let  $X, Y$  be random variables. Then*

- (i)  $\varphi_X(0) = 1$ .
- (ii)  $|\varphi_X| \leq 1$ .
- (iii)  $\forall n \geq 0$ , if  $\mathbb{E}|X|^n < \infty$ , then  $\varphi_X \in C^n(\mathbb{R})$ .
- (iv) If  $X \perp Y$ ,  $\varphi_{X+Y} = \varphi_X \cdot \varphi_Y$ .
- (v)  $\forall t \in \mathbb{R}$ ,  $\overline{\varphi_X(t)} = \varphi_X(-t)$ .
- (vi)  $\forall a, b, t \in \mathbb{R}$ ,  $\varphi_{aX+b}(t) = e^{itb}\varphi_X(at)$ .
- (vii)  $\mathbb{E}(\varphi_X(Y)) = \mathbb{E}(\varphi_Y(X))$ , and if  $X \perp Y$ , they are same as  $\mathbb{E}(\varphi_{XY}(1))$ .

**Example 9.16.** (i)  $X \sim \text{Uniform}(0, 1)$ .  $\varphi_X(t) = \int_0^1 e^{itx} dx = \frac{e^{it}-1}{it}$ .

(ii)  $X \sim \text{Exponential}(\lambda)$ .  $\varphi_X(t) = \int_0^\infty e^{itx} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda-it}$ .

(iii)  $X = X_1 - X_2$ ,  $X_1 \perp X_2$ ,  $X_1, X_2 \sim \text{Exponential}(1)$ .

$$\varphi_X(t) = \varphi_{X_1 - X_2}(t) = \varphi_{X_1}(t) \overline{\varphi_{X_2}(t)} = \frac{1}{1-it} \frac{1}{1+it} = \frac{1}{1+t^2}$$

The density of  $X$  is given by  $f(x) = \frac{1}{2}e^{-|x|}$ , known as the “Laplace density”.

(iv)  $X \sim N(0, 1)$ .  $\varphi_X(t) = e^{-\frac{t^2}{2}}$ .

**1<sup>st</sup> Proof of (iv):** For each  $\theta \in \mathbb{R}$ ,

$$\mathbb{E}(e^{\theta X}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2} + \theta x} dx = \frac{e^{\frac{\theta^2}{2}}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(x-\theta)^2}{2}} dx = e^{\frac{\theta^2}{2}}$$

Now, it can be checked that the functions from  $\mathbb{C}$  to  $\mathbb{C}$  defined  $z \mapsto \mathbb{E}(e^{zX})$  and  $z \mapsto e^{\frac{z^2}{2}}$  are analytic, and the previous calculation shows that they agree on  $\mathbb{R}$ . The identity principle then implies that they must agree everywhere, so in particular  $\varphi_X(t) = \mathbb{E}(e^{itX}) = e^{\frac{(it)^2}{2}} = e^{-\frac{t^2}{2}}$ . ■

**2<sup>nd</sup> Proof of (iv):**  $\phi_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} e^{-\frac{x^2}{2}} dx = e^{-\frac{t^2}{2}}$  by a contour integral argument. ■

**3<sup>rd</sup> Proof of (iv):** By Gaussian integration by parts,  $\mathbb{E}(Xg(X)) = \mathbb{E}(g'(X))$  for all differentiable  $g$  with  $\|g\|_\infty + \|g'\|_\infty < \infty$ . Applying this to  $g(x) = e^{itx}$  yields  $\mathbb{E}(Xe^{itX}) = \mathbb{E}(ite^{itX})$ . Then  $\varphi'_X(t) = \mathbb{E}(iXe^{itX}) = \mathbb{E}(-te^{itX}) = -t\varphi_X(t)$ .  $\varphi_X$  also has initial data  $\varphi_X(0) = 1$ , and so the unique solution to this initial value problem is  $\varphi_X(t) = e^{-\frac{t^2}{2}}$ . ■

**4<sup>th</sup> Proof of (iv):** Let  $(X_i)_{i=1}^\infty$  be iid with distribution  $\mathbb{P}(X_1 = \pm 1) = \frac{1}{2}$ . Then  $\mathbb{E}(X_1) = 0$ ,  $\text{Var}(X_1) = 1$ , and so  $\frac{X_1 + \dots + X_n}{\sqrt{n}} \implies X$  by CLT. This implies  $\varphi_{\frac{X_1 + \dots + X_n}{\sqrt{n}}}(t) \rightarrow \varphi_X(t)$  for all  $t \in \mathbb{R}$ . Next,  $\varphi_{\frac{X_1 + \dots + X_n}{\sqrt{n}}} = \varphi_{X_1 + \dots + X_n}\left(\frac{t}{\sqrt{n}}\right) = \varphi_{X_1}\left(\frac{t}{\sqrt{n}}\right)^n$ . It's an easy check that  $\varphi_{X_1} = \cos$ , so for all  $t \in \mathbb{R}$ ,

$$\varphi_X(t) = \lim_{n \rightarrow \infty} \varphi_{\frac{X_1 + \dots + X_n}{\sqrt{n}}}(t) = \lim_{n \rightarrow \infty} \left( \cos\left(\frac{t}{\sqrt{n}}\right) \right)^n = \lim_{n \rightarrow \infty} \left( 1 - \frac{t^2}{2n} + O\left(\frac{t^4}{n^2}\right) \right)^n = e^{-\frac{t^2}{2}}. \quad \blacksquare$$

**Example 9.17.**  $X \sim N(\mu, \sigma^2)$ .  $\varphi_X(t) = e^{it\mu - \frac{\sigma^2 t^2}{2}}$ .

**Proposition 9.18.** If  $(X_i)_{i=1}^\infty$  is a sequence of random variables with  $X_i \sim N(0, \sigma_i^2)$  for some  $(\sigma_i)_{i=1}^\infty$  bounded away from 0 and  $\infty$ , and  $X_i \xrightarrow{(d)} X$  for some  $X$ , then  $\sigma_i \rightarrow \sigma$  for some  $\sigma$  and  $X \sim N(0, \sigma^2)$ .

**Proof:** Notice that  $e^{-\frac{\sigma_n^2}{2}} = \varphi_{X_n}(1) \rightarrow \varphi_X(1)$ . Since  $(\sigma_i)_i$  is real and bounded away from 0 and  $\infty$ ,  $\varphi_X(1)$  is real and bounded away from 0 and 1. Thus, there exists  $\sigma \in (0, \infty)$  such that  $\phi_X(1) = e^{-\frac{\sigma^2}{2}}$ . This implies  $\sigma_i \rightarrow \sigma$ . Let  $Z \sim N(0, \sigma)$ . Then for every  $t \in \mathbb{R}$ ,

$$\varphi_X(t) = \lim_{n \rightarrow \infty} \varphi_{X_n}(t) = \lim_{n \rightarrow \infty} e^{-\frac{\sigma_n^2 t^2}{2}} = e^{-\frac{\sigma^2 t^2}{2}} = \varphi_Z(t)$$

implying  $X \sim Z$ . ■

## 9.4 Inversion of the Characteristic Function

For any random variable  $X$  with absolutely continuous CDF, let  $\rho_X$  denote its density.

**Lemma 9.19.** Let  $Z$  be a random variable with  $Z \sim N(0, 1)$ , and let  $\sigma > 0$ . Then for any random variable  $X$ ,  $\rho_{X+\sigma Z}(y) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_X(z) e^{-izy} e^{-\frac{\sigma^2 z^2}{2}} dz$ .

*Proof.* By convolution formula, for any random variable  $X$ , the density of  $X + \sigma Z$  at  $x$  is

$$\rho_{X+\sigma Z}(x) = \int_{\mathbb{R}} \sigma^{-1} \phi((x-z)/\sigma) \mathbb{P}_X(dz).$$

Thus we have,

$$\rho_{X+\sigma Z}(0) = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-\frac{z^2}{2\sigma^2}} \mathbb{P}_X(dz) = \frac{1}{\sqrt{2\pi}\sigma} \mathbb{E} \left( \varphi_{\frac{Z}{\sigma}}(X) \right) = \frac{1}{\sqrt{2\pi}\sigma} \mathbb{E} (\varphi_X(Z/\sigma)) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_X(z) e^{-\frac{\sigma^2 z^2}{2}} dz.$$

The desired equality holds for  $y = 0$ . The general case follows by applying this result to the random variable  $X - y$ :

$$\rho_{X+\sigma Z}(y) = \rho_{(X-y)+\sigma Z}(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_{X-y}(z) e^{-\frac{\sigma^2 z^2}{2}} dz = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_X(z) e^{-izy} e^{-\frac{\sigma^2 z^2}{2}} dz.$$
■

**Lemma 9.20.** Let  $X$  be a random variable. If  $\int_{\mathbb{R}} |\varphi_X(t)| dt < \infty$ , then  $X$  has a density  $\rho_X$  and  $\rho_X(y) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_X(t) e^{-ity} dt$ .

*Proof.* Let  $Z$  be a rv independent from  $X$  with  $Z \sim N(0, 1)$ . By the preceding lemma,  $\rho_{X+\sigma Z}(y) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_X(t) e^{-ity} e^{-\frac{\sigma^2 t^2}{2}} dt$ . Then we note that, for any  $g \in C_b(\mathbb{R})$ ,

$$\begin{aligned} \mathbb{E}(g(X + \sigma Z)) &= \int_{\mathbb{R}} g(y) \rho_{X+\sigma Z}(y) dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} \varphi_X(t) e^{-ity} e^{-\frac{\sigma^2 t^2}{2}} dt dy = \frac{1}{2\pi} \int_{\mathbb{R}} g(y) e^{-\frac{\sigma^2 t^2}{2}} \int_{\mathbb{R}} \varphi_X(t) e^{-ity} dy dt. \end{aligned}$$

We also note the fact that as  $\sigma \rightarrow 0$ ,  $X + \sigma Z \xrightarrow{(d)} X$ . Then

$$\begin{aligned} \mathbb{E}(g(X)) &= \lim_{\sigma \rightarrow 0} \mathbb{E}(g(X + \sigma Z)) = \lim_{\sigma \rightarrow 0} \frac{1}{2\pi} \int_{\mathbb{R}} g(y) e^{-\frac{\sigma^2 t^2}{2}} \int_{\mathbb{R}} \varphi_X(t) e^{-ity} dy dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} \varphi_X(t) e^{-ity} dy dt = \frac{1}{2\pi} \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} \varphi_X(t) e^{-ity} dt dy \end{aligned}$$

which implies  $\rho_X(y) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_X(t) e^{-ity} dt$ . ■

**Theorem 9.21 (Inversion Lemma).** For any random variable  $X$  and  $a < b$ ,

$$\lim_{T \rightarrow \infty} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \varphi_X(t) dt = \mathbb{P}(a < X < b) + \frac{1}{2} \mathbb{P}(X \in \{a, b\}).$$

### 9.4.1 Tightness

**Lemma 9.22.**  $\frac{1}{u} \int_{-u}^u (1 - \varphi_X(t)) dt \geq \mathbb{P}(|X| \geq \frac{2}{u})$ .

*Proof.* We have

$$\begin{aligned} \frac{1}{u} \int_{-u}^u (1 - \varphi_X(t)) dt &= \frac{1}{u} \int_{-u}^u (1 - \mathbb{E}(e^{itX})) dt = \mathbb{E} \left( \frac{1}{u} \int_{-u}^u (1 - e^{itX}) dt \right) \\ &= \mathbb{E} \left( 2 - \frac{e^{iuX} - e^{-iuX}}{iuX} \right) \\ &= \mathbb{E} \left( 2 - \frac{2 \sin(uX)}{uX} \right) \\ &\geq 2 \mathbb{E} \left( 1 - \frac{1}{uX} \right) \geq \mathbb{P}(|uX| \geq 2) = \mathbb{P}(|X| \geq 2/u). \end{aligned}$$

**Theorem 9.23 (Lévy's Continuity Theorem).** Let  $(X_n)_n$  be a sequence of random variables and  $\varphi : \mathbb{R} \rightarrow \mathbb{C}$  a function continuous at 0 such that  $\varphi_{X_n}(t) \rightarrow \varphi(t)$  for all  $t \in \mathbb{R}$ . Then there exists a random variable  $X$  such that  $\varphi_X = \varphi$  and  $X_n \Rightarrow X$ .

**Proof:** It is necessary and sufficient to show that  $\{X_n\}$  is tight, or equivalently, that  $\sup_n \mathbb{P}(|X_n| \geq \frac{2}{u}) \rightarrow 0$  as  $u \rightarrow 0$ . By the lemma, it suffices to show  $\sup_n \frac{1}{u} \int_{-u}^u (1 - \varphi_{X_n}(t)) dt \rightarrow 0$ . By DCT we have  $\frac{1}{u} \int_{-u}^u (1 - \varphi_{X_n}(t)) dt \rightarrow \frac{1}{u} \int_{-u}^u (1 - \varphi(t)) dt$ . Since  $\varphi$  is continuous at 0,  $\frac{1}{u} \int_{-u}^u (1 - \varphi(t)) dt \rightarrow 0$  as  $u \rightarrow 0$ . ■