Martingale Convergence Theorems

Example 14.1 (Galton-Watson Process). *Let X be a non-negative integer valued random variable with $\mathbb{E}\, X = \mu$. Let $(X_i^{(j)})$ be a sequence of iid r.v.s with $X_i^{(j)} \stackrel{\mathrm{d}}{=} X$.*

(a) If $Z_0 = 1, Z_1 = X_1^{(1)}$ and

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_i^{(n)} \text{ for } n \geqslant 1,$$

then

$$(M_n = \mu^{-n} \cdot Z_n,\ \mathcal{F}_n = \sigma(X_i^{(j)}, j \leqslant n))$$

is a non-negative martingale. Thus, MGCT implies that

$$M_n = \mu^{-n} \cdot Z_n \xrightarrow{a.s.} M_\infty.$$

If $\mu < 1$, we get $Z_n \stackrel{a.s.}{\to} 0$ and so $\mathbb{P}(\text{finite time extinction}) = 1$.

(b) Suppose $\mathbb{E}\, X = \mu > 1$. Then there exists a unique $\theta \in (0, 1)$ such that $\mathbb{E}\, \theta^X = \theta$ and

$$(Q_n = \theta^{Z_n},\ \mathcal{F}_n = \sigma(X_i^{(j)}, j \leqslant n))$$

is a bounded non-negative martingale. By MGCT $Q_n = \theta^{Z_n} \xrightarrow{a.s.} Q_\infty$ and by DCT $\mathbb{E}\, Q_\infty = \theta$. Since we know that $\mu^{-n} Z_n \xrightarrow{a.s.} M_\infty$ and $\mu > 1$, we have $\lim_{n\to\infty} Z_n \in \{0, \infty\}$ and $\mathbb{P}(Z_n \to 0) = \theta$.

Example 14.2. *For $X \in L^1, X_n := \mathbb{E}(X \mid \mathcal{F}_n)$ is a martingale for any filtration $\mathcal{F}_n$. Moreover, $\mathbb{E}\,|X_n| \leqslant \mathbb{E}\,|X| < \infty$ implies that $X_n \to X_\infty$ a.s., where*

$$X_\infty := \mathbb{E}(X \mid \mathcal{F}_\infty),\ \mathcal{F}_\infty = \sigma(\cup_{n=1}^\infty \mathcal{F}_n).$$

Example 14.3 (MGCT does not imply L^1 convergence). *Consider $\mathcal{F}_n = \sigma(\xi_1 \ldots, \xi_n)$ with i.i.d. r.v.s $xi_i = \pm 1$ with probability $\frac{1}{2}$ and*

$$S_0 = 1,\ S_n = S_{n-1} + \xi_n \text{ for } n \geqslant 1 \text{ and the stopping time } N = \inf\{n \geqslant 0 \mid S_n = 0\}.$$

Then $(M_n = S_{N\wedge n})$ is a non-negative martingale with respect to $(\mathcal{F}_n)$. By MGCT, $M_n \to M_\infty$ a.s. But, $M_\infty = 0$ and $\mathbb{E}(M_n) = \mathbb{E}(M_0) = 1$ for $n \geqslant 1$. Thus M_n does not converge to M_∞ in L^1.

Exercise 14.4. *In the previous example, show that $\mathbb{P}(\sup_n S_{N\wedge n} \geqslant k) = \frac{1}{k}, k \geqslant 1$.*

Exercise 14.5. *If $(M_n, \mathcal{F}_n)_{n\geqslant 1}$ is a martingale such that $M_n \to M_\infty$ a.s. and in L^1, then $M_n = \mathbb{E}(M_\infty | \mathcal{F}_n)$ for all $n \geqslant 1$.*
Hint: *$M_n = \mathbb{E}(M_m \mid \mathcal{F}_n)$ for any $m \geqslant n$.*

From the previous example, we see that we need additional conditions to get L^1 convergence.

14.1 Doob's Decomposition

From the definition of a sub-Martingale $(X_n, \mathcal{F}_n)_{n\geqslant 0}$, it is easy to see that if at every time point n we subtract a positive $\mathcal{F}_n$-measurable r.v. $\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n)$ we can keep the conditional mean zero. Thus, the process $X_n - \sum_{i=0}^{n-1} \mathbb{E}(X_{i+1} - X_i \mid \mathcal{F}_i)$ will be a Martingale. Formally, we have the following.

Theorem 14.6 (Doob's Decomposition). *Let $(X_n, \mathcal{F}_n)_{n\geqslant 0}$ be a sub-martingale. Then there exists a unique decomposition $X_n = M_n + A_n$, where $(M_n, \mathcal{F}_n)_{n\geqslant 0}$ is a martingale with $M_0 \equiv 0$ and $(A_n)_{n\geqslant 0}$ is predictable and non-decreasing.*

Proof. If $X_n = M_n + A_n$, then $\Delta X_n = X_n - X_{n-1} = \Delta M_n + \Delta A_n$. Since ΔA_n is $\mathcal{F}_{n-1}$ measurable and $\mathbb{E}(M_n - M_{n-1}|\mathcal{F}_{n-1}) = 0$, this implies that

$$\mathbb{E}(\Delta X_n|\mathcal{F}_{n-1}) = \Delta A_n, \ \forall n \geqslant 0.$$

Thus, $A_n - A_0 = \sum_{i=0}^{n} \mathbb{E}(X_i - X_{i-1} \mid \mathcal{F}_{i-1}), A_0 = X_0$ and $M_n = X_n - A_n = \sum_{i=1}^{n}(X_i - \mathbb{E}(X_i \mid \mathcal{F}_{i-1}))$. Now, (X_n) is a sub-martingale, hence $\mathbb{E}(X_{n+1}|\mathcal{F}_n) - X_n \geqslant 0$, whoch implies that A_n is non-decreasing. We can easily check that A_n is predictable and M_n is a mean zero martingale. ■

14.2 Martingale Central Limit Theorem

Theorem 14.7. *Let $(M_n, \mathcal{F}_n)_{n\geqslant 1}$ be a mean zero martingale with $s_n^2 := \mathbb{E}\, M_n^2 < \infty, n \geqslant 1$ and $\sigma_n^2 \to \infty$ as $n \to \infty$. Then,*

$$\frac{M_n}{s_n} \xrightarrow{(d)} N(0,1),$$

if,

(i) $s_n^{-2} \sum_{i=1}^{n} \mathbb{E}\left(\Delta_i^2 \mathbb{1}_{|\Delta_i| > \varepsilon s_n}\right) \overset{n\to\infty}{\longrightarrow} 0$ *for all* $\varepsilon > 0$, *where* $\Delta_i = M_i - M_{i-1}$.

(ii) $\frac{1}{s_n^2} \sum_{i=1}^{n} \mathbb{E}(\Delta_i^2 \mid \mathcal{F}_{i-1}) \xrightarrow{\mathbb{P}} 1$.

14.3 Maximal inequalities

Theorem 14.8 (Doob's Maximal Inequality.). *If $(X_n, \mathcal{F}_n)_{n\geqslant 0}$ is a non-negative sub-martingale, and $\bar{X}_n := \max_{1\leqslant i\leqslant n} X_i$, then for all $\lambda > 0, A = \{\bar{X}_n \geqslant \lambda\}$,*

$$\mathbb{P}(A) \leqslant \frac{1}{\lambda}\, \mathbb{E}(X_n \mathbb{1}_A) \leqslant \frac{1}{\lambda}\, \mathbb{E}(X_n).$$

Proof. Define the stopping time $N := \inf\{i \mid X_i \geqslant \lambda\} \wedge n$. Then for $A = \{\bar{X}_n \geqslant \lambda\} = \{X_N \geqslant \lambda\}$ we have

$$\mathbb{P}(A) \leqslant \frac{1}{\lambda}\, \mathbb{E}(X_N \mathbb{1}_A) = \frac{1}{\lambda}(\mathbb{E}\, X_N - \mathbb{E}(X_N \mathbb{1}_{A^C})) \leqslant \frac{1}{\lambda}(\mathbb{E}(X_n) - \mathbb{E}(X_n \mathbb{1}_{A^C})) = \frac{1}{\lambda}\, \mathbb{E}(X_n \mathbb{1}_A),$$

since $N = n$ if $\bar{X}_n < \lambda$. ■

Note that

- If X_n is a sub-martingale, then X_n^+ is a non-negative sub-martingale.
- If X_n is a martingale, then $|X_n|$ is a non-negative sub-martingale.

The above maximal inequality holds for all sub/super/martingales if we replace the inequality by

$$\mathbb{P}(\max_{1\leqslant i\leqslant n} |X_i| \geqslant \lambda) \leqslant \frac{3}{\lambda} \sup_{1\leqslant i\leqslant n} \mathbb{E}(|X_i|).$$

But this is not enough for the next result.

Theorem 14.9 ($\mathbf{L^p}$ Maximal Inequality.). *Let $(X_n, \mathcal{F}_n)_{n\geqslant 0}$ be a non-negative sub-martingale, with $\mathbb{E}\,|X_n|^p < \infty$ for some $p > 1$. Then,*

$$\left\|\bar{X}_n\right\|_p \leqslant \frac{p}{p-1} \left\|X_n\right\|_p.$$

If $p = 1$, then

$$\left\|\bar{X}_n\right\|_1 \leqslant \frac{1 + \mathbb{E}(|X_n| \log^+ X_n)}{1 - e^{-1}}, \text{ provided that } \mathbb{E}(|X_n| \log^+ X_n) < \infty.$$

Here we mention that the constant $p/(p-1)$ is optimal for $p > 1$. If we do not care about the constant, one can directly prove that $\left\|\bar{X}_n\right\|_p \leqslant c \left\|X_n\right\|_p$ for some $c \in (0, \infty)$.

Proof. First we claim that

$$\mathbb{E}(\bar{X}_n^p) = \int_0^\infty p y^{p-1}\, \mathbb{P}(\bar{X}_n \geqslant y) dy.$$

Applying Fubini's, we get

$$\int_0^\infty p y^{p-1}\, \mathbb{P}(\bar{X}_n \geqslant y) dy = \int_\Omega \int_0^\infty p y^{p-1} \mathbb{1}_{\bar{X}_n(\omega)\geqslant y} dy\, \mathbb{P}(d\omega) = \int_\Omega \bar{X}_n(\omega)^p\, \mathbb{P}(d\omega) = \mathbb{E}\,\bar{X}_n^p.$$

Now, by Doob's Maximal Inequality (taking $\lambda = y$), for $p > 1$,

$$\mathbb{E}\,|\bar{X}_n|^p \leqslant \int_0^\infty p y^{p-1} \cdot \frac{1}{y}\, \mathbb{E}(X_n \mathbb{1}_{\bar{X}_n \geqslant y}) dy = \mathbb{E}(X_n \int_0^{\bar{X}_n} p y^{p-2} dy) = \frac{p}{p-1}\, \mathbb{E}(X_n \bar{X}_n^{p-1}).$$

Take $q = \frac{p}{p-1}$, so that $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder's Inequality,

$$\frac{p}{p-1}\, \mathbb{E}(X_n \bar{X}_n^{p-1}) \leqslant \frac{p}{p-1} \|X_n\|_p \|\bar{X}_n^{p-1}\|_q,$$

and

$$\|\bar{X}_n^{p-1}\|_q = (\mathbb{E}(\bar{X}_n^{p-1})^{\frac{p}{p-1}})^{\frac{p-1}{p}} = (\mathbb{E}(\bar{X}_n^p))^{1-\frac{1}{p}}.$$

Thus,

$$\begin{aligned} \mathbb{E}\,|\bar{X}_n|^p &\leqslant \frac{p}{p-1} \|X_n\|_p (\mathbb{E}(\bar{X}_n^p))^{1-\frac{1}{p}} \\ \|\bar{X}_n\|_p = (\mathbb{E}\,|\bar{X}_n|^p)^{\frac{1}{p}} &\leqslant \frac{p}{p-1} \|X_n\|_p. \end{aligned}$$

Here we assumed that $\bar{X}_n^{p-1} \in L^q$. For a more rigorous proof, work with $\bar{X}_n \wedge M$, show that $\|\bar{X}_n \wedge M\|_p \leqslant \frac{p}{p-1} \|X_n\|_p$, and take $M \uparrow \infty$. The proof for $p = 1$ is left as an exercise. ∎

Exercise 14.10. *Show that*

1. *For* $x > 0, y > 0$, $xy \leqslant e^{x-1} + y \log^+ y$. *(Young's Inequality)*
2. $\|\bar{X}_n\| \leqslant \frac{1+\mathbb{E}(X_n \log^+ X_n)}{1-e^{-1}}$.

14.4 L^p Convergence Theorem

Theorem 14.11 (**$\mathbf{L^p}$ Convergence Theorem.**). *Let* $(X_n, \mathcal{F}_n)_{n\geqslant 0}$ *be a non-negative sub-martingale, with*

$$\sup_{n\geqslant 1} \mathbb{E}\,|X_n|^p < \infty \text{ for some } p > 1$$

Then, $X_n \xrightarrow{\mathbf{L^p}} X_\infty$, *in other words,* $\mathbb{E}\,|X_\infty|^p < \infty$, *and* $\mathbb{E}\,|X_n - X_\infty|^p \to 0$.

Proof. By Martingale Convergence Theorem, $X_n \xrightarrow{a.s.} X_\infty$. Thus to show that $X_n \xrightarrow{L^p} X_\infty$, we need to to show two things: 1) $X_\infty \in L^p$ and 2) $X_n \xrightarrow{L^p} X_\infty$.

By L^p Maximal Inequality, we have $\sup_{n\geqslant 1} \left\|\bar{X}_n\right\|_p \leqslant \frac{p}{p-1} \sup_{n\geqslant 1} \|X_n\|_p < \infty$. By Monotone Convergence Theorem, $\bar{X}_n \uparrow \bar{X}_\infty$, so $\mathbb{E}\,\bar{X}_n^p \to \mathbb{E}\,\bar{X}_\infty^p$. Now $X_n \leqslant \bar{X}_\infty$ implies that $X_\infty \leqslant \bar{X}_\infty$ and $|X_n - X_\infty| \leqslant 2\bar{X}_\infty$. Thus, $X_\infty \in L^p$, and by DCT we have, $\mathbb{E}\,|X_n - X_\infty|^p \to 0$. ∎

Corollary 14.12. *If* $(M_n, \mathcal{F}_n)_{n\geqslant 1}$ *is a* L^p *bounded martingale, i.e.* $\sup_{n\geqslant 0} \mathbb{E}\,|M_n|^p < \infty$, *then* $M_n \xrightarrow[L^p]{a.s.} M_\infty$.

Proof. By Martingale Convergence Theorem, $M_n \xrightarrow{a.s.} M_\infty$. Take $X_n = |M_n|$, then X_n is a non-negative sub-martingale, and $\bar{X}_\infty \in L^p$. $\bar{X}_\infty$ is also a bound for M_∞. ∎

Corollary 14.13. *Let* $X \in L^p$, *and* $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots$ *be a filtration. Then* $\mathbb{E}(X \mid \mathcal{F}_n) \xrightarrow[L^p]{a.s.} \mathbb{E}(X \mid \mathcal{F}_\infty)$.

Proof. By MGCT the almost sure part holds, and $\mathbb{E}(|\,\mathbb{E}(X \mid \mathcal{F}_n)|^p) \leqslant \mathbb{E}\,|X|^p < \infty$, thus the L^p convergence follows. ∎

14.5 L^1 Convergence Theorem

The L^p convergence theorem holds for $p > 1$. What happens when $p = 1$?

Theorem 14.14. *If* $(M_n, \mathcal{F}_n)$ *is a martingale, and* $M_n \to M_\infty$ *in* L^1, *then* $M_n = \mathbb{E}(M_\infty \mid \mathcal{F}_n)$.

Proof. For $m \geqslant n$, $M_n = \mathbb{E}(M_m \mid \mathcal{F}_n)$ for all all $m \geqslant n$. Thus $\mathbb{E}\,|\,\mathbb{E}(M_m \mid \mathcal{F}_n) - \mathbb{E}(M_\infty \mid \mathcal{F}_n)| \leqslant \mathbb{E}\,|M_m - M_\infty| \xrightarrow{m\to\infty} 0$, so $\mathbb{E}\,|M_n - \mathbb{E}(M_\infty \mid \mathcal{F}_n)| = 0$. Thus, $M_n = \mathbb{E}(M_\infty \mid \mathcal{F}_n)$. ∎

Definition 14.15 (**Uniform Integrability.**). *A collection of random variables,* $\{X_\lambda : \lambda \in \Lambda\}$, *is uniformly integrable if* $\forall \varepsilon > 0, \exists k > 0$, *such that*

$$\sup_{\lambda\in\Lambda} \mathbb{E}(|X_\lambda| \mathbb{1}_{\{X_\lambda \geqslant k\}}) \leqslant \varepsilon.$$

Corollary 14.16. *If* $(X_\lambda, \lambda \in \Lambda)$ *is u.i., then* $\sup_{\lambda\in\Lambda} \mathbb{E}\,|X_\lambda| < \infty$.

Proof. Fix $\varepsilon = 1$, find k so that $(X_\lambda, \lambda \in \Lambda)$ is u.i.. Then

$$\mathbb{E}\,|X_\lambda| = \mathbb{E}\,|X_\lambda|\mathbb{1}_{|X_\lambda|<k} + \mathbb{E}\,|X_\lambda|\mathbb{1}_{|X_\lambda|\geqslant k} \leqslant k + \varepsilon < \infty.$$ ∎

Exercise 14.17. (i) *If* $|X_n| \leqslant Y$ *for all* n *for some* $Y \in L^1(\mathcal{F})$*, then* (X_n) *is u.i.*

(ii) *If* (X_n) *and* (Y_n) *are both u.i., then so is* $(X_n + Y_n)$.

(iii) *Show that* $X_n \xrightarrow{\mathbf{L^1}} X$ *implies that* (X_n) *is u.i.*

One can check that uniform integrability is just the right condition to go from convergence in probability to L^1-convergence. So that, convergence in probability + uniform integrability = L^1 convergence.

Theorem 14.18. *Suppose* $X_n \to X_\infty$ *in probability and* $(X_n)_{n\geqslant 0}$ *is u.i., then* $X_\infty \in L^1$ *and* $X_n \xrightarrow{L^1} X_\infty$.

Proof. Fix $\varepsilon > 0$, choose k so that $\mathbb{E}(|X_n|\mathbb{1}_{\{|X_n|\geqslant k\}}) \leqslant \varepsilon$ for all n. We know that $X_\infty \in L^1$ by Fatou's Lemma ($\mathbb{E} \liminf |X_n| \leqslant \liminf \mathbb{E}\,|X_n| \leqslant \infty$). So, we can choose k large so that $\mathbb{E}(|X_\infty|\mathbb{1}_{\{|X_\infty|\geqslant k\}}) \leqslant \varepsilon$.

Consider

$$\phi_k(x) = \begin{cases} x & |x| \leqslant k; \\ k & x > k; \\ -k & x < -k. \end{cases}$$

Then $\phi_k(X_n) \to \phi_k(X_\infty)$ in probability and is bounded by k uniformly. By BCT, $\mathbb{E}\,|\phi_k(X_n) - \phi_k(X_\infty)| \xrightarrow{n\to\infty} 0$. Partitioning $\mathbb{E}\,|X_n - X_\infty|$ to two parts, one with either $|X_n|$ or $|X_\infty| \geqslant k$, the other with both $|X_n|$ and $|X_\infty|$ bounded by k, we have

$$\begin{aligned} \mathbb{E}\,|X_n - X_\infty| &\leqslant E(|X_n|\mathbb{1}_{|X_n|\geqslant k} + |X_\infty|\mathbb{1}_{|X_\infty|\geqslant k}) + \mathbb{E}\,|\phi_k(X_n) - \phi_k(X_\infty)| \\ &\leqslant 2\varepsilon + \mathbb{E}\,|\phi_k(X_n) - \phi_k(X_\infty)| \\ \implies \limsup \mathbb{E}\,|X_n - X_\infty| &\leqslant 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, taking $\varepsilon \to 0$, $\mathbb{E}\,|X_n - X_\infty| \to 0$. So $X_n \xrightarrow{L^1} X_\infty$. ∎

Note that

- If (X_n) u.i., then take $\phi \geqslant 0$, $\phi(x) \uparrow \infty$ as $x \uparrow \infty$, then

$$\sup \mathbb{E}(|X_n|\mathbb{1}_{|X_n|\geqslant\lambda}) = \sup \mathbb{E}(|X_n| \cdot \frac{\phi(|X_n|)}{\phi(|X_n|)}\mathbb{1}_{|X_n|\geqslant\lambda}) \leqslant \frac{1}{\phi(\lambda)} \sup \mathbb{E}\,|X_n|\phi(|X_n|).$$

- (X_n) u.i. $\Leftrightarrow \lim_{k\to\infty} \sup_n \mathbb{E}(|X_n|\mathbb{1}_{|X_n|\geqslant k}) = 0$.

Theorem 14.19 ($\mathbf{L^1}$ Convergence Theorem.). *Let* $(M_n, \mathcal{F}_n)$ *be a martingale. Then the following are equivalent:*

(i) $M_n \xrightarrow[L^1]{a.s.} M_\infty$.

(ii) $M_n = \mathbb{E}(M_\infty \mid \mathcal{F}_n), \forall n \geqslant 0$.

(iii) $(M_n)_{n\geqslant 0}$ *is u.i..*

Proof. i) $\implies$ ii) follows from Theorem 23.6. iii) $\implies$ i) follows from MGCT and Theorem 23.9. We'll show ii) $\implies$ iii):

$$\mathbb{E}(|M_n|\mathbb{1}_{|M_n|\geqslant k}) = \mathbb{E}\,|\,\mathbb{E}(M_\infty \mid \mathcal{F}_n)\mathbb{1}_{|M_n|\geqslant k}| \leqslant \mathbb{E}(\mathbb{E}(|M_\infty \mathbb{1}_{|M_n|\geqslant k})| \mid \mathcal{F}_n) = \mathbb{E}(|M_\infty|\mathbb{1}_{|M_n|\geqslant k})).$$

For $\forall n \geqslant 0$, $\mathbb{P}(|M_n| \geqslant k) \leqslant \frac{1}{k}\,\mathbb{E}\,|M_n| \leqslant \frac{1}{k}\,\mathbb{E}\,|M_\infty|$. We claim that $\forall \varepsilon > 0, \exists \delta$ s.t. $\mathbb{P}(A) < \delta$ implies $\mathbb{E}\,|X|\mathbb{1}_A < \varepsilon$. Otherwise, $\exists \varepsilon > 0, \delta_n \downarrow 0$, and $\mathbb{P}(A_n) \leqslant \delta_n$, such that $\mathbb{E}\,|X|\mathbb{1}_{A_n} \geqslant \varepsilon$. (contradiction) By the claim, $\forall \varepsilon > 0$, choose δ satisfying the claim. Choose $k > 0$ such that $\frac{\mathbb{E}\,|M_\infty|}{k} < \delta$, then

$$\mathbb{E}\,|M_n|\mathbb{1}_{|M_n|\geqslant k} \leqslant \mathbb{E}\,|M_\infty|\mathbb{1}_{|M_n|\geqslant k} \leqslant \varepsilon.$$

Thus (M_n) is u.i.. ∎

14.6 Reverse Filtration and Reverse Martingale

Definition 14.20 (Reverse Filtration). *We define a* ***reverse filtration*** *as a sequence of decreasing σ-fields $\mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \mathcal{G}_2 \supseteq \cdots$. For consistency, we will use negative index with $\mathcal{F}_{-n} = \mathcal{G}_n, n \geqslant 0$ so that $\cdots \subseteq \mathcal{F}_{-n} \subseteq \cdots \mathcal{F}_{-2} \subseteq \mathcal{F}_{-1} \subseteq \mathcal{F}_0$ is an increasing sequence of σ-fields.*

Definition 14.21 (Reverse Martingale). *$(M_{-n})_{n\geqslant 0}$ is a* ***reverse martingale*** *w.r.t. the reverse filtration $(\mathcal{F}_{-n})_{n\geqslant 0}$ if*

(i) $M_{-n} \in L^1(\mathcal{F}_{-n}), n \geqslant 0,$

(ii) $\mathbb{E}(M_{-n} \mid \mathcal{F}_{-n-1}) = M_{-n-1}, n \geqslant 0$

Remark*: Clearly $M_{-n} = \mathbb{E}(M_0 \mid \mathcal{F}_{-n})$. Also the definition of sub/super reverse martingale extends in the same way as it does for martingales.*

Theorem 14.22. *Let $(M_{-n}, \mathcal{F}_{-n})_{n\geqslant 0}$ be a reverse martingale. Then*

$$M_{-n} \xrightarrow{a.s.\ and\ L^1} M_{-\infty}.$$

Proof. (i) $(M_{-n})_{n\geqslant 0}$ is u.i.

(ii) Let $U_n(a,b)$ = number of up-crossings of $(M_{-n}, M_{-n+1}, \cdots, M_0)$ of the interval (a,b). By Doob's upcrossing inequality we have $\mathbb{E}\,U_n(a,b) \leqslant \frac{\mathbb{E}\,|M_0^+|+a}{b-a}$. Taking $n \to \infty$, $U_\infty(a,b) < \infty$ *a.e.* $\forall\ a < b$. From here we conclude that M_{-n} converges a.s. to some random variable $M_{-\infty}$.

(iii) $M_{-n} \xrightarrow{a.s.} M_{-\infty}$ and is u.i. This implies $M_{-n} \xrightarrow{\mathbf{L^1}} M_{-\infty}$.

This completes the proof. ∎

Corollary 14.23. *If $M_0 \in L^p$ then $M_{-n} \xrightarrow{\mathbf{L^p}} M_{-\infty}$.*

Corollary 14.24. *$M_{-\infty} = \mathbb{E}(M_0 \mid \mathcal{F}_{-\infty})$ where $\mathcal{F}_{-\infty} = \cap_{n\geqslant 0}\mathcal{F}_{-n}$*

14.6.1 SLLN (using reverse martingale)

Theorem 14.25. *Let $X_1, X_2, \cdots$ be i.i.d. with mean μ. Then*

$$\frac{S_n}{n} \xrightarrow{a.s.} \mu \text{ where } S_n = X_1 + X_2 + \cdots + X_n.$$

Proof. Observe that $\mathcal{F}_{-n} = \sigma(S_n, S_{n+1}, \cdots) = \sigma(S_n, X_{n+1}, X_{n+2}, \cdots), n \geqslant 1$ is a reverse filtration.

Claim: $\mathbb{E}(X_1 \mid \mathcal{F}_{-n}) = \frac{S_n}{n}$.

From symmetry, we have $\mathbb{E}(X_1 \mid \mathcal{F}_{-n}) = \mathbb{E}(X_i \mid \mathcal{F}_{-n})$ for $i = 1, 2, \cdots, n$. To this we need to show that for any $A \in \mathcal{F}_{-n}$,

$$E(X_1 \mathbb{1}_A) = \mathbb{E}(X_i \mathbb{1}_A).$$

Observe that X_1 and X_i are exchangeable conditioned on filtration $\mathcal{F}_{-n}$, *i.e.*,

$$(X_1, S_n, X_{n+1}, \cdots) \stackrel{\mathrm{d}}{=} (X_i, S_n, X_{n+1}, \cdots).$$

Thus, $\mathbb{E}(X_1 \mid \mathcal{F}_{-n}) = \frac{1}{n}\mathbb{E}(\sum_{i=1}^n X_i \mid \mathcal{F}_{-n}) = \frac{S_n}{n}$. Thus $\frac{S_n}{n} \xrightarrow{a.s.}$ & $\xrightarrow{\mathbf{L^1}} M_{-\infty}$ for some $M_{-\infty}$. $M_{-\infty}$ is a constant since it is tail σ-field measurable ($\{\lim \frac{S_n}{n} \in A\}$ is a tail event for all A). Since L^1 convergence preserves mean, $M_{-\infty} = \mu$. ■