Probability Spaces and measure

Probability theory is the mathematical study of randomness. Today we will recall some definitions and results from measure theory. Every experiment or observation results in a outcome or sample point. Let Ω be the **sample space** consisting of all **outcomes**. We will use ω to denote an outcome from the sample space Ω. Let 2^Ω denote the power set of Ω, consisting of all subsets of Ω. First we define field and σ-field.

1.1 Fields and σ-fields

Definition 1.1. *Let $\mathcal{F}$ be a collection of subsets of Ω. $\mathcal{F}$ is called a **field** or **algebra** if the following holds*

(i) $\Omega \in \mathcal{F}$.

(ii) $A \in \mathcal{F}$ *implies that* $A^c \in \mathcal{F}$ *and*

(iii) $A, B \in \mathcal{F}$ *implies that* $A \cup B \in \mathcal{F}$.

Definition 1.2. *Let $\mathcal{F}$ be a collection of subsets of Ω. $\mathcal{F}$ is called a σ**-field** or σ**-algebra** if the following holds*

(i) $\Omega \in \mathcal{F}$.

(ii) $A \in \mathcal{F}$ *implies that* $A^c \in \mathcal{F}$ *and*

(iii) *if* $A_1, A_2, \ldots \in \mathcal{F}$ *is a countable sequence of sets then* $\cup_{i=1}^\infty A_i \in \mathcal{F}$.

Here and in what follows, countable means finite or countably infinite. Clearly every σ-field is a field. Since $(\cap_{i=1}^\infty A_i)^c = \cup_{i=1}^\infty A_i^c$, every σ-field is also closed under countable intersection. The tuple $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is a σ-field is called a **measurable space**. We will use the notation $\uplus$ to **emphasize** disjoint union. Now let us give some examples of field and σ-field.

Example 1.3. $(\Omega, \mathcal{F})$ *with* $\mathcal{F} = \{\emptyset, \Omega\}$. *$\mathcal{F}$ is the smallest σ-field on Ω, called the trivial σ-field.*

Example 1.4. $(\Omega, \mathcal{F})$ *with* $\mathcal{F} = 2^\Omega$. *2^Ω is the largest σ-field on Ω.*

Example 1.5. *Let Ω be any sample space. Let $\mathcal{A}$ be the collection of all countable subsets and co-countable (complement is countable) subsets of Ω. Check that $(\Omega, \mathcal{A})$ is a σ-field. We will call it the countable co-countable σ-field on Ω.*

Example 1.6. *Let $\Omega = \{1, 2, \ldots\}$. Let $\mathcal{A}$ be the collection of all finite subsets and co-finite (complement is finite) subsets of Ω. Check that $(\Omega, \mathcal{A})$ is a field. Note that $\{2\}, \{4\}, \{6\}, \ldots \in \mathcal{A}$ but $\{2, 4, 6, \ldots\} = \uplus_{i=1}^\infty \{2i\} \notin \mathcal{A}$. Thus $\mathcal{A}$ is not a σ-field*

In general σ-field is a complicated object. But it is easy to construct one.

Definition 1.7. *Given a collection $\mathcal{C}$ of subsets of Ω (need not be a field) we define $\sigma(\mathcal{C})$, the σ-field generated by $\mathcal{C}$, as the smallest σ-field containing $\mathcal{C}$. Similarly, we define $\mathcal{A}(\mathcal{C})$, the field generated by $\mathcal{C}$, as the smallest field containing $\mathcal{C}$.*

Lemma 1.8. *For any $\mathcal{C}$, $\sigma(\mathcal{C})$ and $\mathcal{A}(\mathcal{C})$ exist.*

Proof. We will only prove the existence of $\sigma(\mathcal{C})$. The other part is similar. Let Λ be the collection of all σ-fields containing $\mathcal{C}$. Note that Λ is non-empty as 2^{Ω}, the power-set of Ω is in Λ. Define

$$\mathcal{F} = \bigcap_{\mathcal{G} \in \Lambda} \mathcal{G}.$$

Clearly $\mathcal{C} \subseteq \mathcal{F}$ and $\mathcal{F}$ is contained in all σ-fields containing $\mathcal{C}$. We claim that $\mathcal{F}$ is a σ-field (Verify the axioms of σ-field!). This completes the proof. ∎

Even if $\mathcal{C}$ is a "nice" collection of sets, $\sigma(\mathcal{C})$ can become quite complicated. We give some examples.

Example 1.9. *Let Ω be a sample space and $\mathcal{C}$ be the collection of all singleton subsets of Ω. We claim that $\sigma(\mathcal{C})$ is the countable co-countable σ-field on Ω.*

Definition 1.10. *Let Ω be a topological space with $\mathcal{C}$ being the collection of all open subsets of Ω. The σ-field $\mathcal{B}(\Omega) = \sigma(\mathcal{C})$ is called the **Borel** σ-field on Ω.*

We will denote the Borel σ-field of $\mathbb{R}^n$ by $\mathcal{B}^n$. When $n = 1$ we will simply use $\mathcal{B}$.

1.2 Measure

Definition 1.11. *A **measure** is a nonnegative countable additive set function on a measurable space $(\Omega, \mathcal{F})$, i.e.,a function $\mu : \mathcal{F} \to \mathbb{R}$ such that*

(i) *$\mu(A) \geqslant 0$ for all $A \in \mathcal{F}$, and*

(ii) *if $A_i \in \mathcal{F}$ is a countable sequence of disjoint sets (i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$), then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

A **Probability measure** $\mathbb{P}$ is a measure with $\mathbb{P}(\Omega) = 1$. A **Probability space** is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is a σ-field and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$.

We define a pre-measure ν as a nonnegative countable additive set function on $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is only a field, *i.e.,* a function $\mu : \mathcal{A} \to \mathbb{R}$ such that

(i) $\nu(A) \geqslant 0$ for all $A \in \mathcal{A}$, and

(ii) if $A_i \in \mathcal{A}$ is a countable sequence of disjoint sets (*i.e.,* $A_i \cap A_j = \emptyset$ for all $i \neq j$) such that $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$, then

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i).$$

Abusing notation, we will also call a pre-measure as a measure (which will be clear from the context).

It is easy to check that for a measure (or pre-measure) μ we have $\mu(\emptyset) = 0$ (by condition (ii), taking $A_1 = A_2 = \cdots = \emptyset$). First we mention some properties of a measure (also true for pre-measure).

Theorem 1.12. *Let μ be a measure on the measurable space $(\Omega, \mathcal{F})$. Then the following hold.*

(a) *(Monotonicity) If $A \subseteq B$ then $\mu(A) \leqslant \mu(B)$.*

(b) *(Sub-additivity) If $A \subseteq \cup_{i=1}^{\infty} A_i$ then $\mu(A) \leqslant \sum_{i=1}^{\infty} \mu(A_i)$.*

(c) *(Continuity from below) If $A_i \uparrow A$ (i.e., $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \cup_{i=1}^{\infty} A_i$) then $\mu(A_i) \uparrow \mu(A)$.*

(d) *(Continuity from above) If $A_i \downarrow A$ (i.e., $A_1 \supseteq A_2 \supseteq \cdots$ and $A = \cap_{i=1}^{\infty} A_i$) then $\mu(A_i) \downarrow \mu(A)$.*

Before going to the proof, we explain an important **"disjointification"** operation for a sequence of sets that will be used repeatedly. Given a sequence of sets $A_1, A_2, A_3, \ldots$ (which can be arbitrary) if we define

$$B_1 = A_1, B_2 = A_2 \setminus A_1, B_3 = A_3 \setminus (A_1 \cup A_2), B_4 = A_4 \setminus (A_1 \cup A_2 \cup A_3), \ldots$$

then B_i's are disjoint and $\cup_{i=1}^{m} B_i = \cup_{i=1}^{m} A_i$ for all $m \geqslant 1$. Now we proceed to the proof.

Proof. (a) Let $C = B \setminus A$. Clearly, $C \in \mathcal{F}$ and $C \uplus A = B$. Thus

$$\mu(B) = \mu(C \uplus A) = \mu(C) + \mu(A) \geqslant \mu(A).$$

(b) We consider the sequence of sets $B_i = A \cap A_i$ and disjointify it to get C_i such that $A = \cup_{i=1}^{\infty} B_i = \uplus_{i=1}^{\infty} C_i$. Clearly $C_i \subseteq B_i \subseteq A_i$. Thus, using (ii) of the definition of measure and part (a) of this theorem, we get

$$\mu(A) = \mu(\uplus_{i=1}^{\infty} C_i) = \sum_{i=1}^{\infty} \mu(C_i) \leqslant \sum_{i=1}^{\infty} \mu(A_i).$$

(c) Using the disjointification operation on the sequence of sets $A_1, A_2, A_3, \ldots$, we get the disjoint sequence $B_1 = A_1, B_i = A_i \setminus A_{i-1}$. We have $A = \uplus_{i=1}^{\infty} B_i$ and so

$$\mu(A) = \mu(\uplus_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{m \to \infty} \sum_{i=1}^{m} \mu(B_i) = \lim_{m \to \infty} \mu(\uplus_{i=1}^{m} B_i) = \lim_{m \to \infty} \mu(A_m).$$

(d) Note that $A_1 \setminus A_n \uparrow A_1 \setminus A$. Thus part (c) of this theorem implies that $\mu(A_1 \setminus A_n) \uparrow \mu(A_1 \setminus A)$. Now part (a) shows that for $A \subseteq B$, $\mu(B \setminus A) = \mu(B) - \mu(A)$. It follows that $\mu(A_n) \downarrow \mu(A)$. ■

Definition 1.13. *A measure μ is **finite** if $\mu(\Omega) < \infty$.*

Definition 1.14. *A measure μ is $\sigma-$**finite** if $\Omega = \cup_{i=1}^{\infty} A_i$ with $\mu(A_i) < \infty, \forall i$.*

Example 1.15. *Let Ω = set of outcomes if we toss 2 dices = $\{1, 2, \cdots, 6\} \times \{1, 2, \cdots, 6\}$, and $\mathcal{F} = 2^{\Omega}$, then we can define measure $\mu(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{36}$.*

Example 1.16. *Let Ω be a countable set, $\mathcal{F} = 2^{\Omega}$. Take a sequence $p_w \geqslant 0, w \in \Omega$ s.t. $\sum_{w \in \Omega} p_w = 1$, then $\mu(A) = \sum_{w \in A} p_w$ is a probability measure.*

Example 1.17. *Let $\Omega = \mathbb{R}$ and $\mathcal{F}$ be a countable co-countable $\sigma-$field. Define*

$$\mu(A) = \begin{cases} 0, & if\ A\ is\ countable \\ 1, & if\ A\ is\ co-countable. \end{cases}$$

Proof. Clearly, $\mu(A) \geqslant 0$ and $\mu(\Omega) = 1$. Let $A = \uplus_{i=1}^{\infty} A_i$. We consider two separate cases:

Case 1: A is countable. Then each A_i is countable and $\mu(A) = 0 = \sum_{i=1}^{\infty} \mu(A_i)$.

Cbse 2: A is co-countable. If sets X, Y are both co-countable, then $X \cap Y \neq \varnothing$. Otherwise, $X^c \cup Y^c = \Omega$ and the left side is countable while the right side is uncountable. Therefore, when A is co-countable, exactly one of A_i is co-countable, then $\mu(A) = 1 = 1 + 0 + \cdots = \sum_{i=1}^{\infty} \mu(A_i)$.

■

Example 1.18. *Let $\Omega = (0, 1)$ and $\mathcal{F} = \mathcal{B}(\Omega) = \sigma(open\ subsets\ of\ \Omega)$. Define $\mu((a, b]) = b - a, 0 \leqslant a < b < 1$. How to define $\mu(A)$ in general?*

Given a collection of subsets $\mathcal{C}$ one can always construct a field or σ-field containing $\mathcal{C}$. However, given a nonnegative countable additive set function on $\mathcal{C}$ (think about open sets), it might not be possible to extend it as a measure to $\sigma(\mathcal{C})$. However, this is possible under certain condition. We will mention one such condition next.

Definition 1.19. *We define a **semialgebra** $\mathscr{S}$ as a collection of subsets such that*

(i) $\Omega \in \mathscr{S}$,

(ii) *$A, B \in \mathscr{S}$ implies that $A \cap B \in \mathscr{S}$ and*

(iii) *$A \in \mathscr{S}$ implies that A^c is a finite disjoint union of sets in $\mathscr{S}$.*

Note that the collection of sets

$$\mathscr{S} = \{(a, b] \mid -\infty \leqslant a \leqslant b \leqslant \infty\}$$

is a semialgebra.

Lemma 1.20. *let $\mathscr{S}$ be a semialgebra and $\overline{\mathscr{S}} = \mathcal{A}(\mathscr{S}) =$ algebra generated by $\mathscr{S}$. Then $\overline{\mathscr{S}} = \{$finite disjoint union of sets from $\mathscr{S}\}$.*

Proof. Let $\mathcal{C} = \{$finite disjoint union of sets from $\mathscr{S}\}$. Clearly, $\overline{\mathscr{S}} \supseteq \mathcal{C}$. Now we only need to check $\mathcal{C}$ is an algebra. It is easy to see that $\Omega \in \mathcal{C}$.

Claim: $\mathcal{C}$ is closed under finite intersection.
Let $A, B \in \mathcal{C}$, and $A = \uplus_{i=1}^{m} A_i, B = \uplus_{j=1}^{n} B_j$, $A_i, B_j \in \mathscr{S}$. Then $A \cap B = \uplus_{i,j}(A_i \cap B_j) \in \mathcal{C}$.

Claim: $\mathcal{C}$ is closed under complement.
Let $A = \cup_{i=1}^{n} A_i, A_i \in \mathscr{S}$. Then $A^c = \cap_{i=1}^{n} A_i^c, A_i^c \in \mathcal{C}$. By previous claim, we have $A^c \in \mathcal{C}$.

Hence, $\mathcal{C}$ is an algebra. ■

Let μ be a set function on $\mathscr{S}$. We can extend μ to $\overline{\mu}$ on $\overline{\mathscr{S}}$ by $\overline{\mu}(A) = \sum_{i=1}^{n} \mu(A_i)$ where $A = \uplus_{i=1}^{n} A_i$.

Theorem 1.21. *Let $\mathscr{S}$ be a semi-algebra and μ be a set function on $\mathscr{S}$ such that,*

(i) *(positive)* $\mu(A) \geqslant 0$ *for* $A \in \mathscr{S}$,

(ii) *(finitely additive) If* $A = \cup_{i=1}^{n} A_i, A, A_i \in \mathscr{S}$, *then* $\mu(A) = \sum_{i=1}^{n} \mu(A_i)$.

(iii) *(countably subadditive) If* $A \subseteq \cup_{i=1}^{\infty} A_i, A, A_i \in \mathscr{S}$, *then* $\mu(A) \leqslant \sum_{i=1}^{\infty} \mu(A_i)$.

Then μ *can be extended uniquely to a measure* $\overline{\mu}$ *on* $\overline{\mathscr{S}}$ *satisfying* (i), (ii), (iii). *Moreover, if* $\overline{\mu}$ *is* σ*-finite, then* $\overline{\mu}$ *has a unique extension to a measure* $\widehat{\mu}$ *on* $\sigma(\mathscr{S})$.

The proof can be found in the appendix of the textbook.