Math 561 Midterm, Spring 2023

There are 4 questions. Answer as rigorously as possible. Time is 70 minutes and total point is 115 with maximum score 100. You can state and use any results proved in the class.

NAME: _____________________________

1. (30 pts.) Let \( X_0 = (1, 0) \) and \(|\cdot|\) be the Euclidean \( \ell^2 \)-norm. Define \( X_n \in \mathbb{R}^2 \) inductively by declaring that \( X_{n+1} \) is chosen at random from the ball of radius \( |X_n| \) centered at the origin, i.e., \( X_{n+1}/|X_n| \) is uniformly distributed on the ball of radius 1 and independent of \( X_1, X_2, \ldots, X_n \). Prove that,

\[
\frac{1}{n} \log |X_n| \to c \text{ a.s. for some constant } c \in \mathbb{R} \text{ and compute } c.
\]
2. (25 pts.) Answer any **ONE** of the following.

a) Let \((X_n)_{n \geq 1}\) and \((Y_n)_{n \geq 1}\) be two sequences of random variables with \(\sum_{n=1}^{\infty} P(X_n \neq Y_n) < \infty\). Prove that, \(\sum_{n=1}^{\infty} X_n\) converges a.s. if and only if \(\sum_{n=1}^{\infty} Y_n\) converges a.s.

b) Consider the metric \(d(X, Y) = E\left(\frac{|X-Y|}{1+|X-Y|}\right)\) on the set of random variables on a probability space \((\Omega, \mathcal{F}, P)\). Prove that, \(d(X_n, X) \to 0\) as \(n \to \infty\) if and only if \(X_n \to X\) in probability.
3. (30 pts) Let $X, X_1, X_2, \ldots$ be a sequence of nonnegative random variables such that $X_n \to X$ a.s.. Prove that,

$$X_n \to X \text{ in } L^1 \text{ as } n \to \infty \text{ if and only if } \mathbb{E} X_n \to \mathbb{E} X \text{ as } n \to \infty.$$
4. (30 pts.) Answer any **ONE** of the following.

a) Let $X_n$ be a Geometric($1/n$) distributed random variable with

$$P(X_n = k) = \frac{1}{n} \cdot (1 - 1/n)^k \text{ for } k = 0, 1, 2, \ldots,$$

mean $E X_n = n - 1$ and variance $\text{Var}(X_n) = n(n - 1)$. Prove that, $\frac{X_n}{n}$ converges in distribution to the Exponential(1) distribution with distribution function $(1 - e^{-x}) \cdot 1_{x \geq 0}$ as $n \to \infty$.

b) Let $X_n$ be a Poisson($n^2$) distributed random variable with

$$P(X_n = k) = e^{-n^2} \cdot \frac{n^{2k}}{k!} \text{ for } k = 0, 1, 2, \ldots,$$

mean $E X_n = n^2$ and variance $\text{Var}(X_n) = n^2$. Prove that, $\frac{X_n}{n} - n$ converges in distribution to the Normal(0, 1) distribution as $n \to \infty$. 