Math 561 Midterm, Spring 2023

There are 4 questions. Answer as rigorously as possible. Time is 70 minutes and total point is 115 with maximum score 100. **You can state and use any results proved in the class**.

NAME: ______________________________

1. (30 pts.) Let $\boldsymbol{X}_0 = (1, 0)$ and $|\cdot|$ be the Euclidean ℓ^2-norm. Define $\boldsymbol{X}_n \in \mathbb{R}^2$ inductively by declaring that $\boldsymbol{X}_{n+1}$ is chosen at random from the ball of radius $|\boldsymbol{X}_n|$ centered at the origin, *i.e.*, $\boldsymbol{X}_{n+1}/|\boldsymbol{X}_n|$ is uniformly distributed on the ball of radius 1 and independent of $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$. Prove that,

$$\frac{1}{n}\log|\boldsymbol{X}_n| \to c \text{ a.s. for some constant } c \in \mathbb{R} \text{ and compute } c.$$

2. (25 pts.) Answer any **ONE** of the following.

a) Let $(X_n)_{n\geqslant 1}$ and $(Y_n)_{n\geqslant 1}$ be two sequences of random variables with $\sum_{n=1}^{\infty}\mathbb{P}(X_n \neq Y_n) < \infty$. Prove that,

$$\sum_{n=1}^{\infty} X_n \text{ converges a.s. } \textbf{if and only if } \sum_{n=1}^{\infty} Y_n \text{ converges a.s.}$$

b) Consider the metric $\mathrm{d}(X,Y) = \mathbb{E}\left(\frac{|X-Y|}{1+|X-Y|}\right)$ on the set of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that,

$$\mathrm{d}(X_n, X) \to 0 \text{ as } n \to \infty \textbf{ if and only if } X_n \to X \text{ in probability.}$$

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3. (30 pts) Let $X, X_1, X_2, \ldots$ be a sequence of **nonnegative** random variables such that $X_n \to X$ a.s.. Prove that,

$$X_n \to X \text{ in } L^1 \text{ as } n \to \infty \textbf{ if and only if } \mathbb{E}\, X_n \to \mathbb{E}\, X \text{ as } n \to \infty.$$

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4. (30 pts.) Answer any **ONE** of the following.

a) Let X_n be a Geometric$(1/n)$ distributed random variable with

$$\mathbb{P}(X_n = k) = \frac{1}{n} \cdot (1 - 1/n)^k \text{ for } k = 0, 1, 2, \ldots,$$

mean $\mathbb{E}\, X_n = n - 1$ and variance $\mathrm{Var}(X_n) = n(n-1)$. Prove that, $\frac{X_n}{n}$ converges in distribution to the Exponential(1) distribution with distribution function $(1 - e^{-x}) \cdot \mathbb{1}_{x \geqslant 0}$ as $n \to \infty$.

b) Let X_n be a Poisson(n^2) distributed random variable with

$$\mathbb{P}(X_n = k) = e^{-n^2} \cdot \frac{n^{2k}}{k!} \text{ for } k = 0, 1, 2, \ldots,$$

mean $\mathbb{E}\, X_n = n^2$ and variance $\mathrm{Var}(X_n) = n^2$. Prove that, $\frac{X_n}{n} - n$ converges in distribution to the Normal$(0, 1)$ distribution as $n \to \infty$.
