Homework 8

Math 561: Theory of Probability I

Due date: March 23, 2023

Each problem is worth 10 points and only four randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. Show that, for $X_i, i = 1, 2, 3\ldots$ independent (but not identically distributed) random variables with $\mathbb{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = 1$ for all i, $n^{-1/2}\sum_{i=1}^n X_i$ need not converge to N(0,1) in distribution.

2. Lindeberg and Lyapunov impose conditions on moments so that asymptotic normality occurs. However, it is possible to have asymptotic normality even if there are no moments at all. Let X_n assume the values $+1$ and -1 with probability $(1 - 2^{-n})/2$ each and the value 2^k with probability 2^{-k} for $k > n$.

   (a) Show that $\mathbb{E}(X_n^j) = \infty$ for all positive integers j and n.

   (b) Show that $n^{-1/2}\sum_{i=1}^n X_i \implies \mathrm{N}(0, 1)$.
   Hint: Use cutoff and characterization of convergence in distribution w.r.t. almost sure convergence.

3. (**Maxima of Gaussian**) Let $X_1, X_2, \ldots$ be i.i.d. standard normal r.v.s with density $\phi(x) = 1/\sqrt{2\pi}e^{-x^2/2}$ and distribution function $\Phi(x) = \int_{-\infty}^x \phi(t)dt$.

   (i) Prove Mill's ratio approximation:

   $$\frac{1}{x} - \frac{1}{x^3} \leqslant \frac{1 - \Phi(x)}{\phi(x)} \leqslant \frac{1}{x}, \ x > 0,$$

   (ii) Prove that for any $\theta \in \mathbb{R}$, we have

   $$\frac{\mathbb{P}(X_1 > x + \theta/x)}{\mathbb{P}(X_1 > x)} \to e^{-\theta} \text{ as } x \to \infty.$$

   (iii) Define b_n such that $\mathbb{P}(X_1 > b_n) = 1/n$. Show that for $M_n = \max_{1\leqslant i\leqslant n} X_i$ we have, for any $x \in \mathbb{R}$,

   $$\mathbb{P}(b_n(M_n - b_n) \leqslant x) \to \exp(-e^{-x}) \text{ as } n \to \infty.$$

   The distribution with cdf $e^{-e^{-x}}, x \in \mathbb{R}$ is called the Gumble distribution. This result says that the maxima of n iid N(0,1) rvs is concentrated around b_n with fluctuations of order $1/b_n$.
   (iv) Using (i), show that b_n satisfies

   $$b_n\left(b_n - \left[\sqrt{2\log n} - \frac{\log(4\pi\log n)}{2\sqrt{2\log n}}\right]\right) \to 0$$

   as $n \to \infty$.

4. **(Total variation distance.)** Given two probability measure μ, ν on $\mathbb{R}$ we define the *total variation distance* as

   $$d_{\mathrm{TV}}(\mu, \nu) = \sup_A |\mu(A) - \nu(A)| \in [0, 1].$$

   (i) Show that this is indeed a distance and strictly stronger than convergence in distribution.
   (ii) If there exists a countable subset D such that $\mu(D) = \nu(D) = 1$, show that

   $$d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2}\sum_{d\in D} |\mu(\{d\}) - \nu(\{d\})|.$$

(iii) If X_n converges to X in distribution and $\mathbb{P}(X_n \in \mathbb{Z}) = 1, n \geqslant 1$, then

$$d_{\text{TV}}(\mu_n, \mu) \to 0 \text{ as } n \to \infty$$

where μ_n, μ are the distributions of X_n, X, respectively.

5. **(The Lévy Metric.)** (a) Show that

$$\rho(F, G) = \inf\{\varepsilon : F(x - \varepsilon) - \varepsilon \leqslant G(x) \leqslant F(x + \varepsilon) + \varepsilon \text{ for all } x\}$$

defines a metric on the space of distributions and $\rho(F_n, F) \to 0$ if and only if $F_n \xrightarrow{\text{(d)}} F$.

(b) As a corollary prove that, if $F_n \xrightarrow{\text{(d)}} F$ and F is continuous everywhere, then

$$\sup_x |F_n(x) - F(x)| \to 0 \text{ as } n \to \infty.$$

6. **(Converging together lemma.)** (i) If $X_n \Rightarrow X$, $Y_n \Rightarrow c$ where $c > 0$ is a constant and X_n, Y_n are defined on the same probability space, then $X_n + Y_n \Rightarrow X + c$ and $X_n Y_n \Rightarrow cX$. (We assumed $c > 0$ only to make the proof simpler.)

(ii) If $X_1, X_2, \dots$ are i.i.d. with $\mathbb{E}\, X_1 = 0, \text{Var}(X_1) < \infty$, using SLLN, CLT and part (i) prove that

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{X_1^2 + X_2^2 + \cdots + X_n^2}} \Rightarrow \text{N}(0, 1).$$

This is called **Self-normalized Central Limit theorem.**