# Homework 7

Math 561: Theory of Probability I

Due date: March 9, 2023

Each problem is worth 10 points and only five randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. (**SLLN under second moment.**) Prove that if $X_1, X_2, \ldots$ are i.i.d. with $\mathbb{E}(X_1) = 0, \mathrm{Var}(X_1) < \infty$ and $S_n := X_1 + X_2 + \cdots + X_n$ then for any $p > 1/2$ we have

$$\frac{S_n}{n^p} \to 0 \text{ a.s.}$$

   **Hint:** Two ways: Choose an appropriate subsequence $n_k$ such that $S_{n_k}/n_k^p \to 0$ a.s. and $n_{k+1}/n_k \to 1$. Use Kolmogorov's maximal inequality to prove that $n_k^{-p} \max_{n \in [n_k, n_{k+1})} |S_n - S_{n_k}| \to 0$ a.s. Or use Kronecker's lemma with appropriate sum.

2. (**WLLN with infinite mean.**) Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_1 = (-1)^k k) = C/(k^2 \log k), k \geqslant 2$ where $C$ is chosen to make the sum of the probabilities $= 1$. Show that $\mathbb{E}|X_1| = \infty$, but there exists a constant $\mu$ such that $\frac{1}{n}\sum_{i=1}^n X_i \to \mu$ in probability. Follow the proof of standard WLLN.

3. (i) Let $X_i, i = 1, 2, \ldots$ be independent and $S_n := X_1 + X_2 + \cdots + X_n$. Prove that

$$\mathbb{P}(\max_{1 \leqslant j \leqslant n} |S_j| > 2a) \cdot \min_{j \leqslant n} \mathbb{P}(|S_n - S_j| \leqslant a) \leqslant \mathbb{P}(|S_n| > a), \; a > 0.$$

   (ii) Use the setting of the previous problem to show that, $S_n \to S_\infty$ in Probability implies that $D_n := \sup_{k,m \geqslant n} |S_k - S_m| \to 0$ in probability and a.s.; and in particular, $S_n \to S_\infty$ a.s.

   (iii) Moreover, if $X_i$'s are i.i.d. and $n^{-1}S_n$ converges to 0 in Probability, then

$$n^{-1} \max_{i \leqslant n} S_i \to 0 \text{ in Probability.}$$

   ($\mathbb{E}|X_1|$ need not be finite)

4. (**Tail Events.**) Let $(X_i)$ be a sequence of independent random variables, and let $\mathcal{T}$ be its tail $\sigma$-field. Let $S_n = \sum_{i=1}^n X_i$. Let $b_n \uparrow \infty$ be an increasing sequence of real numbers. Which of the following events must be in $\mathcal{T}$? Give a proof sketch or counter-example.

   (i) $\{X_n \to 0\}$, (ii) $\{S_n \text{ converges}\}$, (iii) $\{X_n > b_n \text{ infinitely often}\}$, (iv) $\{S_n > b_n \text{ infinitely often}\}$, (v) $\{S_n > 0 \text{ infinitely often}\}$.

5. (**Directed last passage percolation.**) Consider the lattice quadrant $\{(i,j) : i, j \geqslant 0\}$ with directed edges $(i,j) \to (i+1, j)$ and $(i,j) \to (i, j+1)$. Associate to each edge $e$ an Exponential(1) distributed r.v. $X_e$ with density $e^{-x}\mathbb{1}_{x>0}$ and mean 1, independent for different edges. For each directed path $\Pi$ of length $n$ started at the origin $(0,0)$, let

$$S_\Pi = \sum_{\text{edges } e \text{ in path } \Pi} X_e.$$

   Let $H_n$ be the maximum of $S_\Pi$ over all such paths $\Pi$ of length $n$. It can be shown that $n^{-1}H_n \to c$ a.s., for some constant $c \geqslant 0$.

   (i) Let $\mathcal{P}_n$ be the horizontal directed path with $n$ edges started at the origin. By SLLN, $S_{\mathcal{P}_n}/n \to 1$ a.s. Using the fact that the mgf of Exponential(1) rv is $(1-\theta)^{-1}$ for $\theta < 1$, prove that

$$\log \mathbb{P}(S_{\mathcal{P}_n} \geqslant an) \leqslant n(\log a + 1 - a) \text{ when } a > 1.$$

   (ii) Using (i), give explicit nontrivial upper and lower bounds on $c$.

6. Let $F, F_n, n \geqslant 1$ be a sequence of CDF's on $\mathbb{R}$ such that $F_n \to F$ in distribution. Prove that $F_n^{-1}(y) \to F^{-1}(y)$ as $n \to \infty$ for all $y \in (0,1)$ except a countable number of points, where

$$F^{-1}(y) := \sup\{u \mid F(u) < y\} = \inf\{u \mid F(u) \geqslant y\}.$$