Homework 6

Math 561: Theory of Probability I

Due date: March 2, 2023

Each problem is worth 10 points and only five randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. (Chebyshev’s other inequality.) Let $f, g: \mathbb{R} \to \mathbb{R}$ be two nondecreasing bounded functions. Prove that for any random variable $X$, we have

$$E(f(X)g(X)) \geq E(f(X)) \cdot E(g(X)).$$

In other words, $f(X)$ and $g(X)$ are positively correlated.

**Hint:** Use an independent copy $Y$ of $X$.

2. (L²-LLN for weakly dependent rvs.) Let $(X_i)_{i \geq 1}$ be r.v.s with $E(X_i) = 0$ and $E(X_i X_j) = r(j - i), 1 \leq i \leq j$, where $(r(n))_{n \geq 0}$ is a deterministic sequence with $r(n) \to 0$ as $n \to \infty$. Prove that

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to 0 \text{ in Probability.}$$

3. Prove that the following are equivalent.

   (i) $X_n \to X$ a.s.
   (ii) For $M_n := \max_{k \geq n} |X_k - X|$, we have $M_n \to 0$ in probability.
   (iii) There exists $\varepsilon_n \downarrow 0$ such that $P(|X_n - X| > \varepsilon_n \text{ i.o.}) = 0$.
   (iv) For every $\varepsilon > 0$ we have $P(|X_n - X| > \varepsilon \text{ i.o.}) = 0$.

4. Let $X_n, n \geq 1$ be i.i.d. r.v.s and $\alpha > 0$ be fixed. Let $M_n = \max\{X_1, X_2, \ldots, X_n\}$.

   (i) Prove that $n^{-1/\alpha} X_n \to 0$ a.s. if and only if $E|X_1|^\alpha < \infty$.
   (ii) Prove that $n^{-1/\alpha} M_n \to 0$ a.s. if and only if $E(\max\{X_1, 0\})^\alpha < \infty$.
   (iii) Find necessary and sufficient conditions for (i), (ii), when a.s. convergence is replaced by convergence in probability.

5. Let $X_1, X_2, \ldots$ be i.i.d. random variables such that

$$\frac{1}{x^\alpha} \log P(X_1 > \beta x) \to -1 \text{ as } x \to \infty$$

for some $\alpha, \beta > 0$. Show that for $c_n := \beta(\log n)^{1/\alpha}$ we have

$$\limsup_{n \to \infty} \frac{X_n}{c_n} = 1 \text{ a.s.,}$$

i.e.,

$$P \left( \frac{X_n}{c_n} \leq 1 + \varepsilon \text{ eventually} \right) = P \left( \frac{X_n}{c_n} \geq 1 - \varepsilon \text{ i.o.} \right) = 1 \text{ for all } \varepsilon > 0.$$
6. (St. Petersburg Paradox.) Let $X_1, X_2, \ldots$ be i.i.d. positive r.v.s with

$$P(X_1 = 2^k) = 2^{-k}, \quad k \geq 1.$$ 

One can think of $X_1$ as the payoff in a gambling where you get $2^k$ if the first head appear in the $k$-th toss (using an unbiased coin). The paradox here is that $E(X_1) = \infty$, but you clearly wouldn’t pay an arbitrary large amount to play this game. Show that,

$$\frac{S_n}{nm_n} \to 1 \text{ in Probability}$$

where $S_n = X_1 + X_2 + \cdots + X_n$ and $m_n$ is a sequence of integers satisfying

$$n2^{-m_n} \to 0 \text{ and } n2^{-m_n} \cdot m_n^2 \to \infty.$$ 

Conclude that,

$$\frac{S_n}{n \log_2 n} \to 1 \text{ in Probability}.$$ 

So, a fair price for playing the game $n$ times should be $\log_2 n$ per play. For more details see [https://plato.stanford.edu/entries/paradox-stpetersburg/](https://plato.stanford.edu/entries/paradox-stpetersburg/)

**Hint:** Constant cutoff won’t work! Choose an appropriate cutoff and do a careful analysis. Use the fact that $P(S_n \in A) \leq P(\max_{1 \leq i \leq n} X_i > a) + P(S_n \in A, \max_{1 \leq i \leq n} X_i \leq a)$ for any $a > 0, A \in \mathcal{B}$. 