Homework 6

Math 561: Theory of Probability I

Due date: March 2, 2023

Each problem is worth 10 points and only five randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. (**Chebyshev's other inequality.**) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two nondecreasing bounded functions. Prove that for any random variable X, we have

$$\mathbb{E}(f(X)g(X)) \geqslant \mathbb{E}(f(X)) \cdot \mathbb{E}(g(X)).$$

In other words, $f(X)$ and $g(X)$ are positively correlated.

Hint: Use an independent copy Y of X.

2. **($\mathbf{L^2}$-LLN for weakly dependent rvs.)** Let $(X_i)_{i\geqslant 1}$ be r.v.s with $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i X_j) = r(j-i), 1 \leqslant i \leqslant j$, where $(r(n))_{n\geqslant 0}$ is a deterministic sequence with $r(n) \to 0$ as $n \to \infty$. Prove that

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to 0 \text{ in Probability.}$$

3. Prove that the following are equivalent.

   (i) $X_n \to X$ a.s.

   (ii) For $M_n := \max_{k\geqslant n} |X_k - X|$, we have $M_n \to 0$ in probability.

   (iii) There exists $\varepsilon_n \downarrow 0$ such that $\mathbb{P}(|X_n - X| > \varepsilon_n \text{ i.o.}) = 0$.

   (iv) For every $\varepsilon > 0$ we have $\mathbb{P}(|X_n - X| > \varepsilon \text{ i.o.}) = 0$.

4. Let $X_n, n \geqslant 1$ be i.i.d. r.v.s and $\alpha > 0$ be fixed. Let $M_n = \max\{X_1, X_2, \dots, X_n\}$.

   (i) Prove that $n^{-1/\alpha} X_n \to 0$ a.s. if and only if $\mathbb{E}\,|X_1|^\alpha < \infty$.

   (ii) Prove that $n^{-1/\alpha} M_n \to 0$ a.s. if and only if $\mathbb{E}(\max\{X_1, 0\})^\alpha < \infty$.

   (iii) Find necessary and sufficient conditions for (i), (ii), when a.s. convergence is replaced by convergence in probability.

5. Let $X_1, X_2, \dots$ be i.i.d. random variables such that

$$\frac{1}{x^\alpha} \log \mathbb{P}(X_1 > \beta x) \to -1 \text{ as } x \to \infty$$

for some $\alpha, \beta > 0$. Show that for $c_n := \beta(\log n)^{1/\alpha}$ we have

$$\limsup_{n\to\infty} \frac{X_n}{c_n} = 1 \text{ a.s.,}$$

i.e.,

$$\mathbb{P}\left(\frac{X_n}{c_n} \leqslant 1 + \varepsilon \text{ eventually}\right) = \mathbb{P}\left(\frac{X_n}{c_n} \geqslant 1 - \varepsilon \text{ i.o.}\right) = 1 \text{ for all } \varepsilon > 0.$$

6. **(St. Petersburg Paradox.)** Let $X_1, X_2, \ldots$ be i.i.d. positive r.v.s with

$$\mathbb{P}(X_1 = 2^k) = 2^{-k}, \quad k \geqslant 1.$$

One can think of X_1 as the payoff in a gambling where you get \$$2^k$ if the first head appear in the k-th toss (using an unbiased coin). The paradox here is that $\mathbb{E}(X_1) = \infty$, but you clearly wouldn't pay an arbitrary large amount to play this game. Show that,

$$\frac{S_n}{nm_n} \to 1 \text{ in Probability}$$

where $S_n = X_1 + X_2 + \cdots + X_n$ and m_n is a sequence of integers satisfying

$$n2^{-m_n} \to 0 \text{ and } n2^{-m_n} \cdot m_n^2 \to \infty.$$

Conclude that,

$$\frac{S_n}{n \log_2 n} \to 1 \text{ in Probability.}$$

So, a fair price for playing the game n times should be \$$\log_2 n$ per play. For more details see https://plato.stanford.edu/entries/paradox-stpetersburg/.

Hint: Constant cutoff won't work! Choose an appropriate cutoff and do a careful analysis. Use the fact that $\mathbb{P}(S_n \in A) \leqslant \mathbb{P}(\max_{1 \leqslant i \leqslant n} X_i > a) + \mathbb{P}(S_n \in A, \max_{1 \leqslant i \leqslant n} X_i \leqslant a)$ for any $a > 0, A \in \mathcal{B}$.