Homework 5

Math 561: Theory of Probability I

Due date: February 23, 2023

Each problem is worth 10 points and only five randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. Prove that the following are equivalent.
   (i) $X_n \to X$ in Probability.
   (ii) $\mathbb{E}\min(|X_n - X|, 1) \to 0$.
   (iii) There exists $\varepsilon_n \downarrow 0$ such that $\mathbb{P}(|X_n - X| \geqslant \varepsilon_n) \leqslant \varepsilon_n$.

2. (**Independence & Orthogonality**.) Recall that, $L^2(\Omega, \mathcal{F}, \mathbb{P}) := \{X : \Omega \to \mathbb{R} \mid X \text{ is a r.v. on } (\Omega, \mathcal{F}) \text{ with } \mathbb{E}(X^2) < \infty\}$ is a Hilbert space under the inner product $\langle X, Y\rangle := \mathbb{E}(XY)$. Let $\mathcal{G}$ be a sub σ-field. Define
$$L_0^2(\Omega, \mathcal{G}, \mathbb{P}) := \{X : \Omega \to \mathbb{R} \mid X \text{ is a r.v. on } (\Omega, \mathcal{G}) \text{ with } \mathbb{E}(X) = 0, \mathbb{E}(X^2) < \infty\}.$$
Show that $\mathcal{F}_1, \mathcal{F}_2$ are independent iff $L_0^2(\Omega, \mathcal{F}_1, \mathbb{P}),\ L_0^2(\Omega, \mathcal{F}_2, \mathbb{P})$ are orthogonal subspaces of $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

3. **(Improved Borel–Cantelli Lemma.)** Suppose the events $\{A_n, n \geqslant 1\}$ satisfy $\mathbb{P}(A_n) \to 0$ as $n \to \infty$ and
$$\sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty.$$
Prove that
$$\mathbb{P}(A_n \text{ occurs infinitely often}) = 0.$$

4. Let X be a r.v. and g be a differentiable function on $\mathbb{R}$ such that $g(0) = 0, \mathbb{E}\,|g(X)| < \infty$. Using Fubini's theorem for $\mathbb{P}_X \otimes$ Lebesgue to prove that
$$\mathbb{E}(g(X)) = \int_{\mathbb{R}} \operatorname{sgn}(x) g'(x)\, \mathbb{P}(\operatorname{sgn}(x) X \geqslant |x|) dx$$
where $\operatorname{sgn}(x) = \mathbb{1}_{x>0} - \mathbb{1}_{x<0}$ is the signum function. When $g(x) \equiv x$ is the identity function and $X \geqslant 0$, we get back the result
$$\mathbb{E}(X) = \int_0^{\infty} \mathbb{P}(X \geqslant x) dx.$$

5. Let $\mathbb{P}$ be a probability measure on $(\mathbb{R}, \mathcal{B})$ and F be the corresponding distribution function $F(x) := \mathbb{P}((-\infty, x])$, $x \in \mathbb{R}$. Prove that, for any $c \in \mathbb{R}$ we have
$$\int_{\mathbb{R}} (F(x+c) - F(x))\, dx = c.$$

6. We define the Poisson(λ) distribution as the distribution of X with
$$\mathbb{P}(X = i) = \frac{\lambda^i e^{-\lambda}}{i!}, i \geqslant 0.$$
Suppose X has Poisson(λ) distribution and Y has Poisson(μ) distribution with $\mu \geqslant \lambda$.
(i) Using exponential Markov's inequality $\mathbb{P}(X \geqslant x) \leqslant \inf_{t>0} e^{-tx}\, \mathbb{E}\, e^{tX}$, prove that
$$\mathbb{P}(X \geqslant Y) \leqslant \exp(-(\sqrt{\lambda} - \sqrt{\mu})^2)$$

if X and Y are independent. Use the fact that, if X has Poisson(λ) distribution then $\mathbb{E}(e^{tX}) = e^{\lambda(e^t-1)}, t \in \mathbb{R}$.

(ii) Find constants $A, c \in (0, \infty)$, not depending on λ, such that, without assuming independence,

$$\mathbb{P}(X \geqslant Y) \leqslant A \exp(-c(\sqrt{\lambda} - \sqrt{\mu})^2).$$

7. A collection of r.v.s $X_i, i = 1, 2, \ldots, n$ is called *k-wise independent* if $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ are independent for any choice of $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. Clearly, independence implies k-wise independent for all $k \geqslant 2$. The goal of the exercise is to show that, 2-wise (pairwise) independence does not imply 3-wise independence.

   Let $q \geqslant 3$ be a prime integer. Let X_0 and X_1 be independent random variables that are uniformly distributed on $\mathbb{F}_q := \{0, 1, \ldots, q-1\}$. For $0 \leqslant n < q$, let

$$Z_n := (X_0 + nX_1) \mod q.$$

   Show that $Z_0, Z_1, \ldots, Z_{q-1}$ are pairwise independent, *i.e.,* each pair is independent, but not 3-wise independent.

   Extra/hard: Prove the result for all $k \geqslant 2$, *i.e.,* k-wise independence does not imply $(k+1)$-wise independence. Use $X_0, X_1, \ldots, X_{k-1}$ i.i.d. uniformly distributed over $\mathbb{F}_q$ and define $Z_n = (\sum_{i=0}^{k-1} n^i X_i) \mod q$ for $0 \leqslant n < q$.