Homework 4

Math 561: Theory of Probability I

Due date: February 16, 2023

Each problem is worth 10 points and only five randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. (i) Prove that Markov's inequality $\mathbb{P}(X \geqslant t) \leqslant \mathbb{E}(X)/t$ is sharp for fixed $t > 0$, *i.e.,* there is a non-negative random variable X such that $\mathbb{P}(X \geqslant t) = \mathbb{E}(X)/t$.

   (ii) (One-sided Chebyshev inequality) Suppose that $E(X) = 0$, $\mathrm{Var}(X) = \sigma^2 < \infty$ and $a > 0$. Prove that
   $$\mathbb{P}(X \geqslant a) \leqslant \frac{\sigma^2}{\sigma^2 + a^2}$$
   and there is a r.v. X for which equality holds.

2. The variance of a r.v. X is defined as
   $$\mathrm{Var}(X) := \mathbb{E}(X - \mathbb{E}\, X)^2.$$
   (i) For a counting r.v. $X = \sum_{i=1}^{n} 1_{A_i}$, give a formula for the variance of X in terms of the probabilities $\mathbb{P}(A_i)$ and $\mathbb{P}(A_i A_j), i < j$.

   (ii) If k balls are put at random into n boxes, what is the variance of X = number of empty boxes?

3. (i) (Second moment inequality) Suppose that $X \geqslant 0$ and $\mathbb{E}\, X^2 < \infty$. Apply the Cauchy-Schwarz inequality to $X\mathbf{1}_{X>0}$ and conclude
   $$\mathbb{P}(X > 0) \geqslant \frac{(\mathbb{E}\, X)^2}{\mathbb{E}\, X^2}.$$
   (ii) Generalize the technique of (iii) to find a nontrivial lower bound for
   $$\mathbb{P}(X > a\, \mathbb{E}\, X),\ a \in (0, 1).$$

4. (i) For a non-negative r.v. X prove that
   $$\inf_{p\in\{0,1,2,\ldots\}} \mathbb{E}\, X^p \leqslant \inf_{t\geqslant 0} e^{-t}\, \mathbb{E}\, e^{tX}.$$
   Remark: By Markov's inequality we have $\mathbb{P}(X \geqslant 1) \leqslant \inf_{p\geqslant 0} \mathbb{E}\, X^p$. Thus, optimal moment bounds gives better probability estimate than exponential Markov's inequality.

   (ii) Show that, strict inequality holds in the above inequality iff $\mathbb{E}\, X < 1, \mathbb{P}(X > 1) > 0$.

5. Show that:

   (i) if $y > 0$ is fixed,
   $$\inf\{\mathbb{P}(|X| > y) \mid \mathbb{E}(X) = 0, \mathbb{E}(X^2) = 1\} = 0.$$
   (ii) if $y \geqslant t > 0$ are fixed,
   $$\inf\{\mathbb{P}(|X + t| > y) \mid \mathbb{E}(X) = 0, \mathbb{E}(X^2) = 1\} = 0.$$

6. (i) Let X be non-negative integer valued r.v. Show that
   $$\mathbb{E}(X) = \sum_{k=1}^{\infty} \mathbb{P}(X \geqslant k).$$
   (ii) Using, (i) show that for any non-negative integrable r.v. X we have
   $$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X \geqslant x)dx.$$