Homework 3

Math 561: Theory of Probability I

Due date: February 9, 2023

Each problem is worth 10 points and only five randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. Let $X : ([0,1], \mathcal{B}[0,1], \mathbb{P} = \text{Lebesgue measure}) \to (\mathbb{R}, \mathcal{B})$ be an integrable random variable.

   (i) Use the definition of the expectation to conclude that given $\varepsilon > 0$, there is a simple rv $\phi = \sum_{i=1}^{k} b_i \mathbf{1}_{A_i}$ such that $\mathbb{E}\,|X - \phi| < \varepsilon$.

   (ii) Show that, given $A \in \mathcal{B}$ and $\varepsilon > 0$, there is a continuous function h such that $\mathbb{E}\,|\mathbf{1}_A - h| < \varepsilon$.

   (iii) conclude that given $\varepsilon > 0$, there exists a continuous function $Y : [0,1] \to \mathbb{R}$ such that $\mathbb{E}\,|X - Y| \leqslant \varepsilon$.

2. Consider the probability space $(\mathbb{R}, \mathcal{B}, \mathbb{P})$, where $\mathbb{P}$ has density $\rho(x)$ w.r.t. the Lebesgue measure, *i.e.,*

$$\mathbb{P}(A) = \int_A \rho(x)dx \text{ for all } A \in \mathcal{B},$$

   for a non-negative measurable function ρ. Show that, for an integrable r.v. X on $(\mathbb{R}, \mathcal{B}, \mathbb{P})$ we have

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} X(x)\rho(x)dx.$$

   Hint: Use the four step procedure.

3. Let $X_n, n \geqslant 1$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that, $X_n \to X$ a.s., if $\mathbb{P}(\lim_{n\to\infty} X_n \text{ exists and } = X) = 1$; and $X_n \to X$ in Probability, if for any $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$.
   (a) Show that, $X_n \to X$ a.s. implies that $X_n \to X$ in Probability.
   (b) Show that the converse is not true, *i.e.,* give an example where $X_n \to X$ in Probability but $X_n \not\to X$ a.s.

4. Use the monotone convergence theorem to prove the following.
   (i) If $X_n \geqslant 0, X_n \downarrow X$ a.s. and $\mathbb{E}(X_n) < \infty$ for some n then $\mathbb{E}(X_n) \downarrow \mathbb{E}(X)$.
   (ii) If $\mathbb{E}\,|X| < \infty$ then $\mathbb{E}(|X| \cdot \mathbb{1}_{|X|>n}) \to 0$ as $n \to \infty$.
   (iii) If $\mathbb{E}(|X_1|) < \infty$ and $X_n \uparrow X$ a.s. then either $\mathbb{E}(X_n) \uparrow \mathbb{E}(X) < \infty$ or else $\mathbb{E}(X_n) \uparrow \infty$ and $\mathbb{E}(|X|) = \infty$.
   (iv) If X takes values in the non-negative integers then

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geqslant n).$$

5. (a) Let X be a r.v. with $||X||_\infty < \infty$. Prove that $||X||_p \to ||X||_\infty$ as $p \to \infty$.

   (b) Prove that, for any rvs X, Y we have $\mathbb{E}\,|XY| \leqslant ||X||_\infty ||Y||_1$.

6. Given a rv $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ for $p \in [1, \infty)$ and $\varepsilon > 0$, show that there exists a simple rv ϕ such that $||X - \phi||_p < \varepsilon$.

7. Prove that $L^1(\Omega, \mathcal{F}, \mathbb{P}) = \{X \text{ r.v. on } (\Omega, \mathcal{F}, \mathbb{P}) \text{ with } \mathbb{E}\,|X| < \infty\}$ is a Banach space (complete normed vector space) under the norm $\|X\| = \mathbb{E}\,|X|$, *i.e.,* check that $\|\cdot\|$ is a norm on the vector space $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and any Cauchy sequence in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ has a limit.