Homework 2

Math 561/Stat 551: Theory of Probability I

Due date: February 2, 2023

Each problem is worth 10 points and only five randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. Let $C = C([0,1])$ be the collection of continuous real functions on $[0,1]$, equipped with the metric $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. It is well-known that C is complete and separable under the metric d. Let $\mathcal{B}(C)$ be the Borel σ-algebra on C.

   Show that, the collection $\mathcal{S}$ of sets of the form

   $$\cap_{i=1}^{k}\{f \in C \mid f(t_i) \in A_i\}$$

   for some $k \geqslant 1, 0 \leqslant t_1 < t_2 < \cdots < t_k \leqslant 1, A_1, A_2, \ldots, A_k \in \mathcal{B}(\mathbb{R})$, is a semi-algebra and generates $\mathcal{B}(C)$.

   You can use the fact that for fixed $t \in [0,1]$, the evaluation map $f \mapsto \pi_t(f) := f(t)$ is continuous from C to $\mathbb{R}$ and $d(f,g) = \sup_{x\in\mathbb{Q}\cap[0,1]} |f(x) - g(x)|$.

2. Show that a distribution function F has at most countably many discontinuities, *i.e.,* the set $D(F) := \{x \mid F$ is discontinuous at $x\}$ is countable. Can $D(F)$ be dense in $\mathbb{R}$? Explain.

3. (Patching two r.v.s) Suppose X, Y, Z are three random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $A \in \mathcal{F}$. Show that the function

   $$W(\omega) = \begin{cases} X(\omega) + Y(\omega) & \text{if } \omega \in A, \\ -Y(\omega) + Z(\omega) & \text{if } \omega \in A^c \end{cases}$$

   is a random variable.

4. Show that the Borel σ-field on $\mathbb{R}^d$ is the smallest σ-field that makes all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ measurable.

5. A function $f : \mathbb{R}^d \to \mathbb{R}$ is lower semicontinuous (l.s.c.) if $\liminf_{y\to x} f(y) \geqslant f(x)$ for all x. A function is upper semicontinuous (u.s.c.) if $\limsup_{y\to x} f(y) \leqslant f(x)$ for all x. Show that, if f is l.s.c. or u.s.c., then f is measurable.

6. Let $f_n : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}), n \geqslant 1$ be a sequence of measurable functions such that $f_n(\omega) \to f(\omega)$ as $n \to \infty$ for every $\omega \in \Omega$. Show that $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ is also measurable, *i.e.,* the class of measurable functions is closed under point-wise limit.

7. Show that if $F(x) = \mathbb{P}(X \leqslant x)$ is continuous for all x then $Y = F(X)$ has a uniform distribution on $(0,1)$, *i.e.,* $\mathbb{P}(Y \leqslant y) = y$ for $0 \leqslant y \leqslant 1$. Why is the condition "$F$ is continuous" necessary?