Homework 12

Math 561: Theory of Probability I

Due date: April 27, 2023

Each problem is worth 10 points and only four randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. Let \((X_n)_{n \geq 0}\) and \((Y_n)_{n \geq 0}\) be martingales w.r.t. the same filtration \((\mathcal{F}_n)_{n \geq 0}\) with \(E(X_n^2 + Y_n^2) < \infty\) for all \(n\). Show that

\[
E(X_n Y_n) = E(X_0 Y_0) + \sum_{m=1}^{n} E(X_m - X_{m-1})(Y_m - Y_{m-1}).
\]

2. Let \(\xi_1, \xi_2, \ldots\) be i.i.d. ±1 with probability 1/2. Define \(S_n = \xi_1 + \xi_2 + \ldots + \xi_n\) and \(T = \inf\{n \mid S_n \geq 1\}\).
(a) Prove that, \(M_n = \theta^{S_n}/(\cosh \theta)^n, n \geq 1\) is a martingale w.r.t. \(\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n), n \geq 1\) where \(\cosh \theta = (e^{\theta} + e^{-\theta})/2\).
(b) Use Wald’s identities to argue that,

\[
E(z^T) = \frac{1 - \sqrt{1 - z^2}}{z} \text{ for all } z \in (0, 1).
\]
(c) From here (comparing coefficients) conclude that

\[
P(T = 2k - 1) = \frac{(2k)!}{2^{2k}k!^2(2k-1)}, k \geq 1.
\]

Note: By Stirling’s approximation \(n! \approx \sqrt{2\pi e^{-n}n^{n+1/2}}\) we have \(P(T = 2k - 1) \approx \frac{1}{2^{2k}k!^2(2k-1)}\) as \(k \to \infty\).

3. (Krickeberg’s decomposition for martingales.) Let \((X_n)\) be a martingale with \(\sup_n E|X_n| < \infty\). Show that there is a representation \(X_n = Y_n - Z_n\) where \((Y_n)\) and \((Z_n)\) are non-negative martingales.

Hint: We have \(X_n = X_n^+ - X_n^-\), but \(X_n^+\) in general is not a martingale, but a sub-martingale. Now use the fact that, \(X_n = E(X_{m+n}^+ \mid \mathcal{F}_n) - E(X_{m+n}^- \mid \mathcal{F}_n)\) and \(E(X_{m+n}^+ \mid \mathcal{F}_n)\) is monotone for all \(m \geq 0\).

4. Let \(S_n, n \geq 0\) be the symmetric random walk (SRW) on \(\mathbb{Z}\) with \(S_0 = 0, S_n = S_{n-1} + \xi_n\) where \(\xi_1, \xi_2, \ldots\) are i.i.d. with \(P(\xi_1 = +1) = 1/2 = P(\xi_1 = -1)\). Let \(N = \inf\{n \geq 0 \mid S_n = -1\}\) be the hitting time of \(-1\).
(i) Prove that \(P(\sup_{n \geq 0} S_{N \wedge n} \geq m) = \frac{1}{m+1}\) for all \(m \geq 0\).
(ii) Use (i) to show that \(N < \infty\) a.s. This proves that \(S_{N \wedge n} \to -1\) a.s.

Note: This implies that for a SRW starting at 0, it will come back to \(-1\) a.s. and in between the path will go up to height \(H\) having distribution \(P(H \geq m) = 1/(m+1), m \geq 0\). In fact the whole piecewise linear path, after appropriate scaling, will converge to a continuous path, called Brownian excursion (to be proved in 562).
5. Let $S_n$ be the total assets of an insurance company at the end of year $n$, starting with an initial asset $S_0 = s_0 > 0$. Suppose that in year $n$ the company receives premiums of $c$ and pays claims totaling $\xi_n$, where $\xi_n$ are independent with Normal($\mu, \sigma^2$) distribution, where $0 < \mu < c$. So that

$$S_n = s_0 + \sum_{i=1}^{n} (c - \xi_i), \quad n \geq 0.$$ 

The company is ruined if its assets fall to 0 or below, i.e., $S_n \leq 0$, at some time. Show that

$$\mathbb{P}(\text{Ruin}) \leq \exp\left(-2(e-\mu)s_0/\sigma^2\right).$$

**Hint:** Construct an appropriate Likelihood Ratio Martingale using the fact that

$$\mathbb{E}e^{\theta \xi_n} = e^{\mu \theta + \sigma^2 \theta^2 / 2}$$

and an appropriate stopping time.