Homework 12

Math 561: Theory of Probability I

Due date: April 27, 2023

Each problem is worth 10 points and only four randomly chosen problems will be graded if there are more than 5 problems. Please indicate whom you worked with, it will not affect your grade in any way.

1. Let $(X_n)_{n\geqslant 0}$ and $(Y_n)_{n\geqslant 0}$ be martingales w.r.t. the same filtration $(\mathcal{F}_n)_{n\geqslant 0}$ with $\mathbb{E}(X_n^2+Y_n^2)<\infty$ for all n. Show that

$$\mathbb{E}(X_nY_n)=\mathbb{E}(X_0Y_0)+\sum_{m=1}^{n}\mathbb{E}(X_m-X_{m-1})(Y_m-Y_{m-1}).$$

2. Let $\xi_1,\xi_2,\ldots$ be i.i.d. ±1 with probability $1/2$. Define $S_n=\xi_1+\xi_2+\cdots+\xi_n$ and $T=\inf\{n\mid S_n\geqslant 1\}$.
(a) Prove that, $M_n=e^{\theta S_n}/(\cosh\theta)^n, n\geqslant 1$ is a martingale w.r.t. $\mathcal{F}_n=\sigma(\xi_1,\ldots,\xi_n), n\geqslant 1$ where $\cosh\theta=(e^\theta+e^{-\theta})/2$.
(b) Use Wald's identities to argue that,

$$\mathbb{E}(z^T)=\frac{1-\sqrt{1-z^2}}{z}\text{ for all }z\in(0,1).$$

(c) From here (comparing coefficients) conclude that

$$\mathbb{P}(T=2k-1)=\frac{(2k)!}{2^{2k}k!^2(2k-1)},k\geqslant 1.$$

Note: By Stirling's approximation $n!\approx\sqrt{2\pi}e^{-n}n^{n+1/2}$ we have $\mathbb{P}(T=2k-1)\approx\frac{1}{2\sqrt{\pi k^3}}$ as $k\to\infty$.

3. (**Krickeberg's decomposition for martingales.**) Let (X_n) be a martingale with $\sup_n\mathbb{E}|X_n|<\infty$. Show that there is a representation $X_n=Y_n-Z_n$ where (Y_n) and (Z_n) are non-negative martingales.

Hint: We have $X_n=X_n^+-X_n^-$, but X_n^+ in general is not a martingale, but a sub-martingale. Now use the fact that, $X_n=\mathbb{E}(X_{m+n}^+\mid\mathcal{F}_n)-\mathbb{E}(X_{m+n}^-\mid\mathcal{F}_n)$ and $\mathbb{E}(X_{m+n}^+\mid\mathcal{F}_n)$ is monotone for all $m\geqslant 0$.

4. Let $S_n, n\geqslant 0$ be the symmetric random walk (SRW) on $\mathbb{Z}$ with $S_0=0, S_n=S_{n-1}+\xi_n$ where $\xi_1,\xi_2,\ldots$ are i.i.d. with $\mathbb{P}(\xi_1=+1)=1/2=\mathbb{P}(\xi_1=-1)$. Let $N=\inf\{n\geqslant 0\mid S_n=-1\}$ be the hitting time of -1.

(i) Prove that $\mathbb{P}(\sup_{n\geqslant 0}S_{N\wedge n}\geqslant m)=\frac{1}{m+1}$ for all $m\geqslant 0$.

(ii) Use (i) to show that $N<\infty$ a.s. This proves that $S_{N\wedge n}\to -1$ a.s.

Note: This implies that for a SRW starting at 0, it will come back to -1 a.s. and in between the path will go upto height H having distribution $\mathbb{P}(H\geqslant m)=1/(m+1), m\geqslant 0$. In fact the whole piecewise linear path, after appropriate scaling, will converge to a continuous path, called Brownian excursion (to be proved in 562).

5. Let S_n be the total assets of an insurance company at the end of year n, starting with an initial asset $S_0 = s_0 > 0$. Suppose that in year n the company receives premiums of c and pays claims totaling ξ_n, where ξ_n are independent with Normal(μ, σ^2) distribution, where $0 < \mu < c$. So that

$$S_n = s_0 + \sum_{i=1}^{n} (c - \xi_i), \quad n \geqslant 0.$$

The company is ruined if its assets fall to 0 or below, *i.e.*, $S_n \leqslant 0$, at some time. Show that

$$\mathbb{P}(\text{Ruin}) \leqslant \exp(-2(c-\mu)s_0/\sigma^2).$$

Hint: Construct an appropriate Likelihood Ratio Martingale using the fact that

$$\mathbb{E}\, e^{\theta \xi_n} = e^{\mu\theta + \sigma^2\theta^2/2}$$

and an appropriate stopping time.